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Mathematical Methods 2024 Meet The Assessors Presentation Exam 2

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Presenter Contact Details

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MAV Meet The Assessors 2024 MM

- The MAV has made the 2024 MAV Solutions to 2023 VCAA Mathematical Methods exams resource available as downloadable files.

- The files can be downloaded easily from the Thinkific platform

Scaling 2023

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- Mean 34.3 (33.9 in 2022) and SD 8.5 (8.4 in 2022)
- 20 (21)
- 25 (28)
- 30 (35)
- 35 (41) Up 1 from last year
- 40 (46) Up 1 from last year
- 45 (49)
- 50 (51)

Changes 2024

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- Four alternatives only in the Multiple Choice
- New Formula Sheet
- VCAA will be releasing
  - The Marking Guide and
  - Fully Worked Solutions

Multiple Choice 2023

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- 77% answered Question 1 correctly.

Question	Answer	Question	Answer
1	E	11	E
2	A	12	E
3	E	13	C
4	B	14	D
5	D	15	A
6	D	16	B
7	C	17	B
8	C	18	E
9	C	19	D
10	B	20	A

E Question 1 (B 16) 77%

Question 1

The amplitude, A, and the period, P, of the function $f(x) = -\frac{1}{2}\sin(3x + 2\pi)$ are

A. $A = -\frac{1}{2},\ P = \frac{\pi}{3}$

B. $A = -\frac{1}{2},\ P = \frac{2\pi}{3}$

C. $A = -\frac{1}{2},\ P = \frac{3\pi}{2}$

D. $A = \frac{1}{2},\ P = \frac{\pi}{3}$

E. $A = \frac{1}{2},\ P = \frac{2\pi}{3}$

$f(x) = -\frac{1}{2}\sin(3x + 2\pi)$

The amplitude is $\left|-\frac{1}{2}\right| = \frac{1}{2}$.

The period is $\frac{2\pi}{3}$.

B Question 4 (A,D, 15,14) 55%

Question 4

Consider the system of simultaneous linear equations below containing the parameter k

$$kx + 5y = k + 5$$
$$4x + (k+1)y = 0$$

The value(s) of k for which the system of equations has infinite solutions are

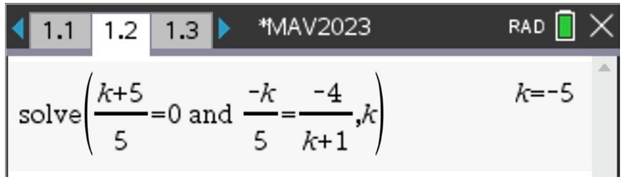
A. $k \in \{-5, 4\}$

B. $k \in \{-5\}$

C. $k \in \{4\}$

D. $k \in R \setminus \{-5, 4\}$

E. $k \in R \setminus \{-5\}$



$kx + 5y = k + 5$, $y = -\frac{k}{5}x + \frac{k+5}{5}$

$4x + (k+1)y = 0$, $y = \frac{-4}{k+1}x$

For infinite solutions

$\frac{k+5}{5} = 0$ and $\frac{-k}{5} = \frac{-4}{k+1}$

Hence $k = -5$.

A Question 2 (B 29) 53%

Question 2

For the parabola with equation $y = ax^2 + 2bx + c$, where $a, b, c \in R$, the equation of the axis of symmetry is

A. $x = -\dfrac{b}{a}$

B. $x = -\dfrac{b}{2a}$

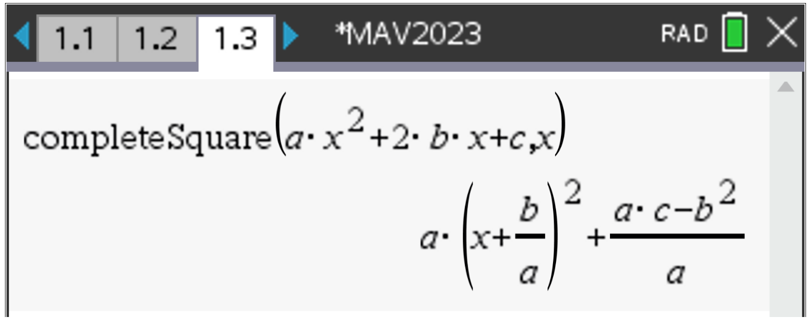
C. $y = c$

D. $x = \dfrac{b}{a}$

E. $x = \dfrac{b}{2a}$

$$y = ax^2 + 2bx + c$$

The equation of the axis of symmetry is $x = \dfrac{-2b}{2a} = -\dfrac{b}{a}$.



C Question 13 (B 22) 52%

Question 13

The following algorithm applies Newton's method using a **For** loop with 3 iterations.

```
Inputs: f(x), a function of x
        df(x), the derivative of f(x)
        x0, an initial estimate

Define newton(f(x), df(x), x0)
    For i from 1 to 3
        If df(x0) = 0 Then
            Return "Error: Division by zero"
        Else
            x0 ← x0 − f(x0) ÷ df(x0)
    EndFor
    Return x0
```

The **Return** value of the function `newton(x³ + 3x − 3, 3x² + 3, 1)` is closest to

A. 0.83333

B. 0.81785

C. 0.81773

D. 1

E. 3

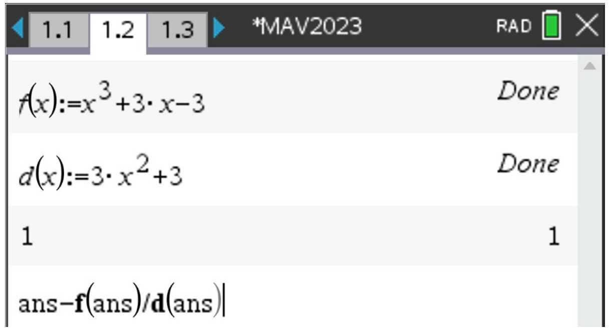
$f(x) = x^3 + 3x - 3,\ f'(x) = 3x^2 + 3,\ x_0 = 1$

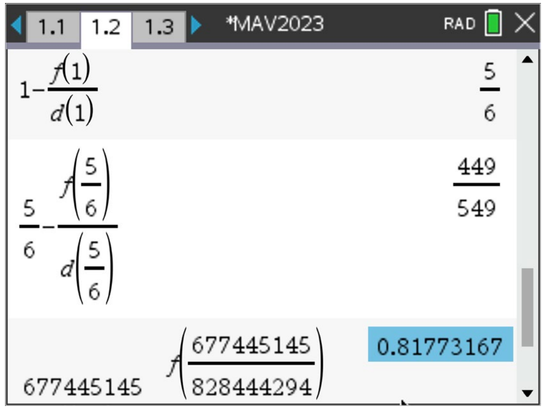
Newton's method

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_0 = 1,\ x_1 = \frac{5}{6},\ x_2 = \frac{449}{549},\ x_3 = 0.81773...$$

After three iterations $x_3 = 0.81773$ correct to five decimal places.





D Question 6 (C 41) 49%

Question 6

Suppose that $\int_3^{10} f(x)dx = C$ and $\int_7^{10} f(x)dx = D$. The value of $\int_7^{3} f(x)dx$ is

A. $C + D$

B. $C + D - 3$

C. $C - D$

D. $D - C$

E. $CD - 3$

$$\int_3^{10} f(x)dx = C \text{ and } \int_7^{10} f(x)dx = D.$$

$$\int_3^{10} f(x)dx = \int_3^{7} f(x)dx + \int_7^{10} f(x)dx$$

$$C = \int_3^{7} f(x)dx + D$$

$$C - D = \int_3^{7} f(x)dx$$

$$\int_7^{3} f(x)dx = D - C$$

C Question 8 (D 16) 49%

Question 8

A box contains n green balls and m red balls. A ball is selected at random, and its colour is noted. The ball is then replaced in the box.

In 8 such selections, where $n \neq m$, what is the probability that a green ball is selected at least once?

A. $8\left(\frac{n}{n+m}\right)\left(\frac{m}{n+m}\right)^7$

B. $1-\left(\frac{n}{n+m}\right)^8$

C. $1-\left(\frac{m}{n+m}\right)^8$

D. $1-\left(\frac{n}{n+m}\right)\left(\frac{m}{n+m}\right)^7$

E. $1-8\left(\frac{n}{n+m}\right)\left(\frac{m}{n+m}\right)^7$

Let G represent a green ball being selected.

$\Pr(G \geq 1) = 1 - \Pr(G = 0)$

$= 1 - \left(\frac{m}{n+m}\right)^8$

E Question 3 (A 22) 47%

Question 3

Two functions, p and q, are continuous over their domains, which are $[-2, 3)$ and $(-1, 5]$, respectively.

The domain of the sum function $p + q$ is

A. $[-2, 5]$

B. $[-2, -1) \cup (3, 5]$

C. $[-2, -1) \cup (-1, 3) \cup (3, 5]$

D. $[-1, 3]$

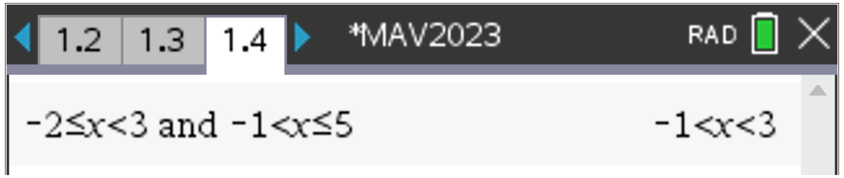
E. $(-1, 3)$

The domain of p is $[-2,3)$ and the domain of q is $(-1,5]$.

The domain of the sum function $p+q$ is the intersection of the two domains.

$$[-2,3)\cap(-1,5]$$

$$=(-1,3)$$



C Question 9 (D 31) 42%

Question 9

The function f is given by

$$f(x)=\begin{cases}\tan\left(\frac{x}{2}\right) & 4\le x<2\pi\\ \sin(ax) & 2\pi\le x\le 8\end{cases}$$

The value of a for which f is continuous and smooth at $x = 2\pi$ is

A. -2

B. $-\frac{\pi}{2}$

C. $-\frac{1}{2}$

D. $\frac{1}{2}$

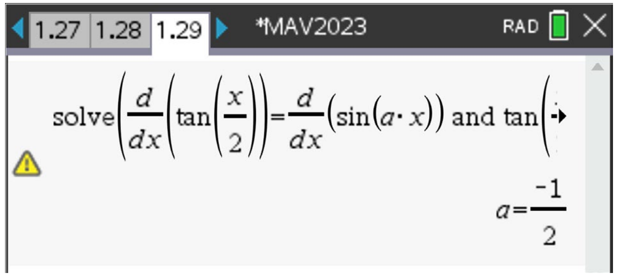
E. 2

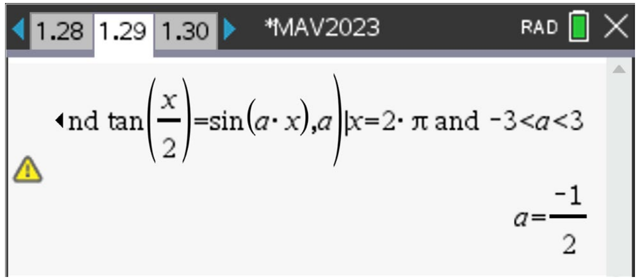
For f to be continuous at $x=2\pi$, $\tan\left(\frac{x}{2}\right)=\sin(ax)$.

For f to be smooth at $x=2\pi$, $\frac{d}{dx}\tan\left(\frac{x}{2}\right)=\frac{d}{dx}\sin(ax)$.

So solve $\tan\left(\frac{x}{2}\right)=\sin(ax)$ and $\frac{d}{dx}\tan\left(\frac{x}{2}\right)=\frac{d}{dx}\sin(ax)$ for a.

$a=-\frac{1}{2}$





D Question 19 (A 27) 32%

Question 19

Find all values of k, such that the equation $x^2 + (4k+3)x + 4k^2 - \frac{9}{4} = 0$ has two real solutions for x, one positive and one negative.

A. $k > -\frac{3}{4}$

B. $k \geq -\frac{3}{4}$

C. $k > \frac{3}{4}$

D. $-\frac{3}{4} < k < \frac{3}{4}$

E. $k < -\frac{3}{4}$ or $k > \frac{3}{4}$

$$x^2 + (4k+3)x + 4k^2 - \frac{9}{4} = 0$$

Solve $\Delta = (4k+3)^2 - 4\left(4k^2 - \frac{9}{4}\right) > 0$ for k for two unique solutions.

$$k > -\frac{3}{4}$$

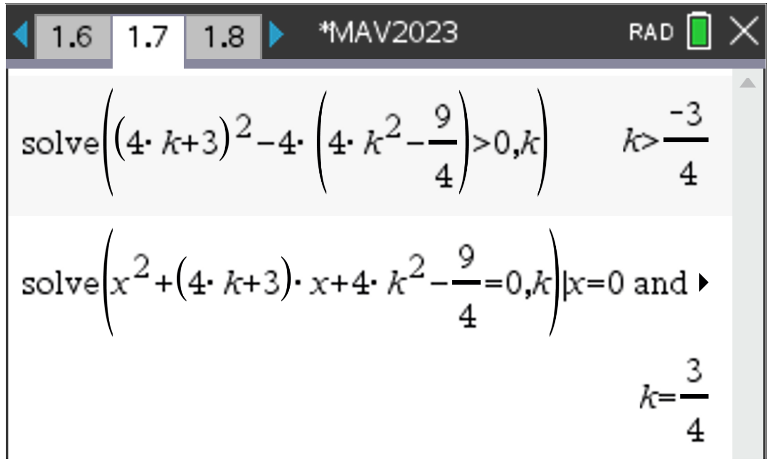
One solution has to be positive and the other negative.

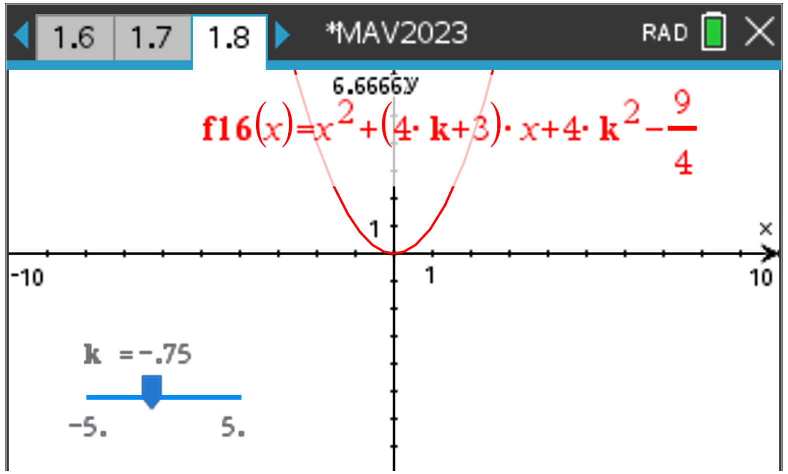
Solve $x^2 + (4k+3)x + 4k^2 - \frac{9}{4} = 0$ for k, when $x = 0$ and $k > -\frac{3}{4}$.

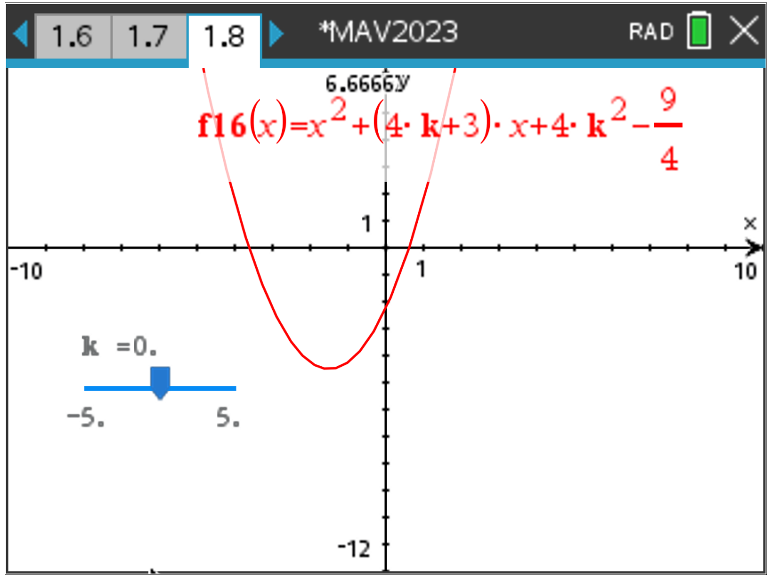
$$k = \frac{3}{4}$$

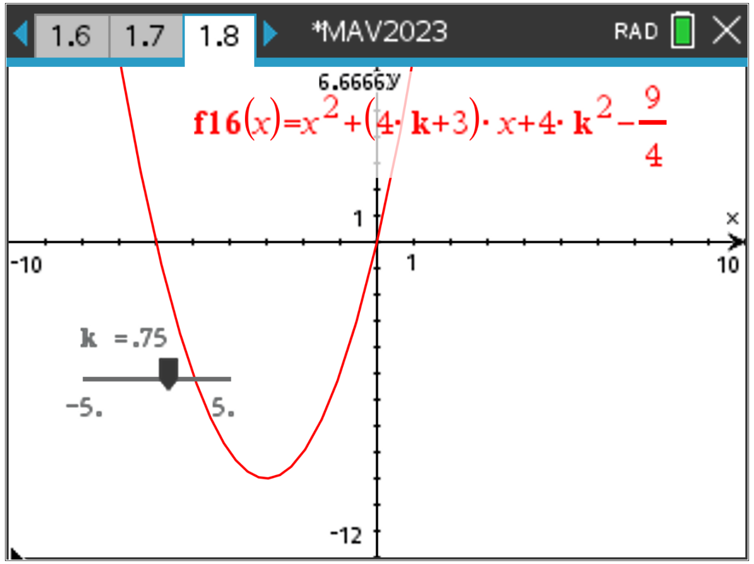
$$-\frac{3}{4} < k < \frac{3}{4}$$

D Question 19 (A 27) 32%









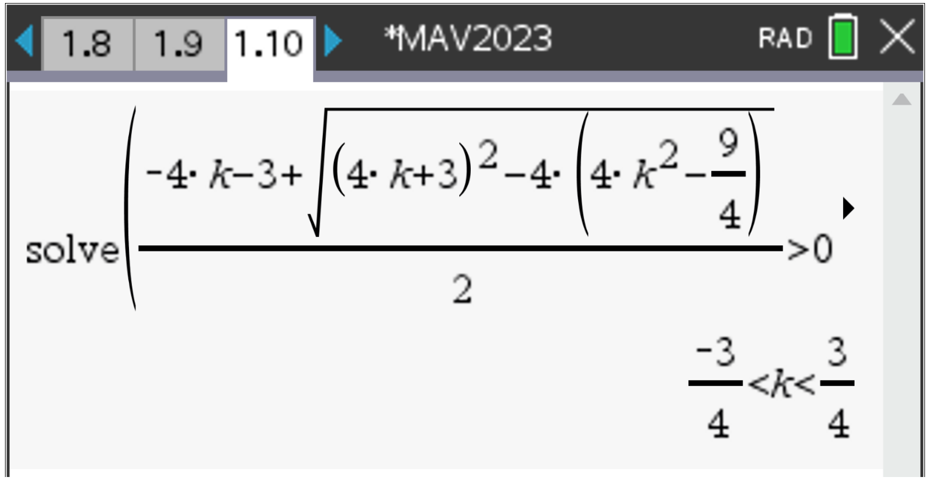
D Question 19 (A 27) 32%

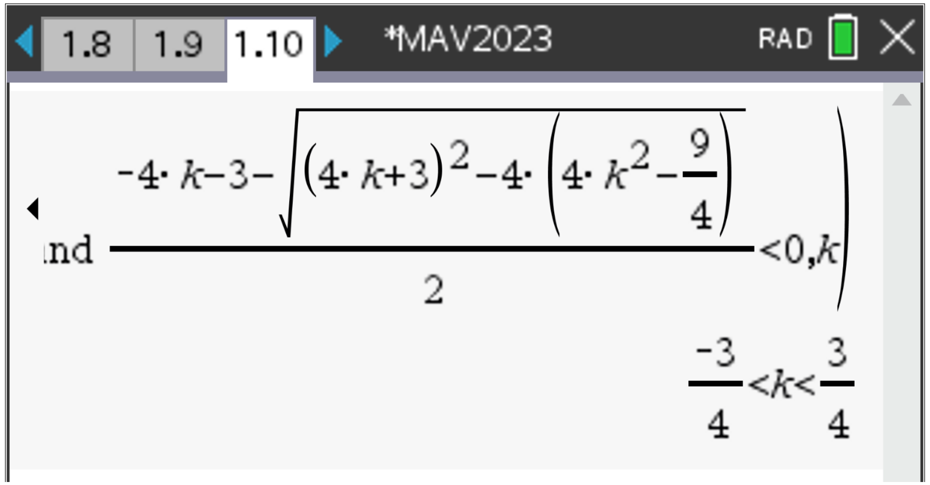
OR

Use the quadratic formula and solve $\dfrac{-b+\sqrt{b^2-4ac}}{2a}>0$ and $\dfrac{-b-\sqrt{b^2-4ac}}{2a}<0$.

So solve $\dfrac{-4k-3+\sqrt{(4k+3)^2-4\left(4k^2-\frac{9}{4}\right)}}{2}>0$ and $\dfrac{-4k-3-\sqrt{(4k+3)^2-4\left(4k^2-\frac{9}{4}\right)}}{2}<0$ for k.

$-\dfrac{3}{4}<k<\dfrac{3}{4}$





A Question 20 (C 26) 30%

Question 20

Let $f(x)=\log_e\left(x+\frac{1}{\sqrt{2}}\right)$.

Let $g(x)=\sin(x)$ where $x \in (-\infty, 5)$.

The largest interval of x values for which $(f \circ g)(x)$ and $(g \circ f)(x)$ both exist is

A. $\left(-\frac{1}{\sqrt{2}}, \frac{5\pi}{4}\right)$

B. $\left[-\frac{1}{\sqrt{2}}, \frac{5\pi}{4}\right)$

C. $\left(-\frac{\pi}{4}, \frac{5\pi}{4}\right)$

D. $\left[-\frac{\pi}{4}, \frac{5\pi}{4}\right]$

E. $\left[-\frac{\pi}{4}, -\frac{1}{\sqrt{2}}\right]$

$$f(x)=\log_e\left(x+\frac{1}{\sqrt{2}}\right),\ g(x)=\sin(x) \text{ where } x\in(-\infty,5)$$

$$(f\circ g)(x)=\log_e\left(\sin(x)+\frac{1}{\sqrt{2}}\right)$$

Hence, $\sin(x)+\frac{1}{\sqrt{2}}>0$.

Solve $\sin(x)=-\frac{1}{\sqrt{2}}$, $x=\ldots-\frac{\pi}{4},\frac{5\pi}{4}$ as $x<5$

Hence $x\in\left(-\frac{\pi}{4}+2\pi k,\frac{5\pi}{4}+2\pi k\right)$, $k\in Z^-\cup\{0\}$.

$$(g\circ f)(x)=\sin\left(\log_e\left(x+\frac{1}{\sqrt{2}}\right)\right)$$

Solve $\log_e\left(x+\frac{1}{\sqrt{2}}\right)<5$, $x=e^5-\frac{1}{\sqrt{2}}$

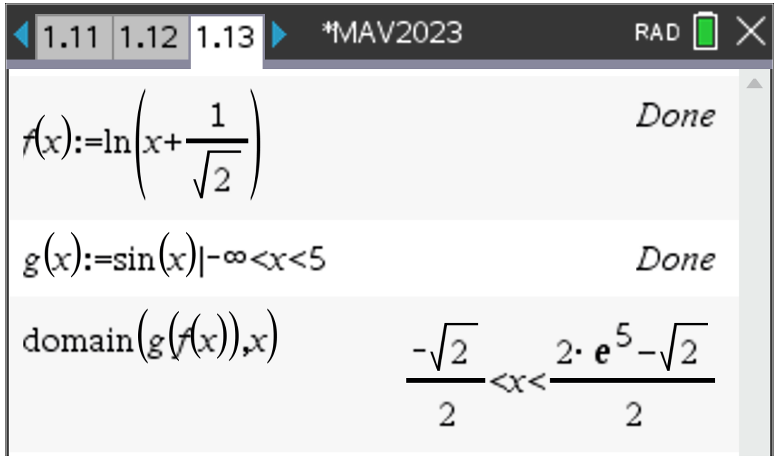
Hence $x\in\left(-\frac{1}{\sqrt{2}},e^5-\frac{1}{\sqrt{2}}\right)$.

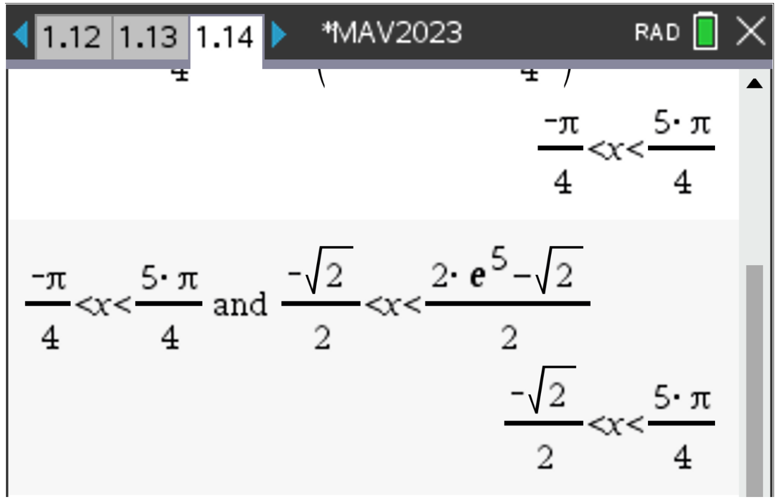
The largest interval of x values for which either $(f\circ g)(x)$ or $(g\circ f)(x)$ exist is

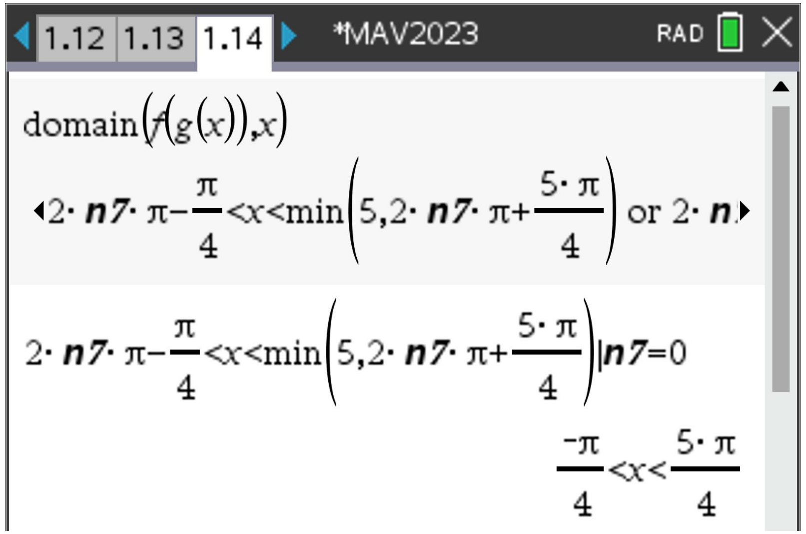
$$\left(-\frac{\pi}{4}+2\pi k,\frac{5\pi}{4}+2\pi k\right)\cap\left(-\frac{1}{\sqrt{2}},e^5-\frac{1}{\sqrt{2}}\right),\ k\in Z^-\cup\{0\}$$

$$=\left(-\frac{1}{\sqrt{2}},\frac{5\pi}{4}\right)$$

A Question 20 (C 26) 30%







E Question 18 (A,D, 22,26) 29%

Question 18

Consider the function $f : [-a\pi, a\pi] \rightarrow R, f(x) = \sin(ax)$, where a is a positive integer.

The number of local minima in the graph of $y = f(x)$ is always equal to

A. 2

B. 4

C. a

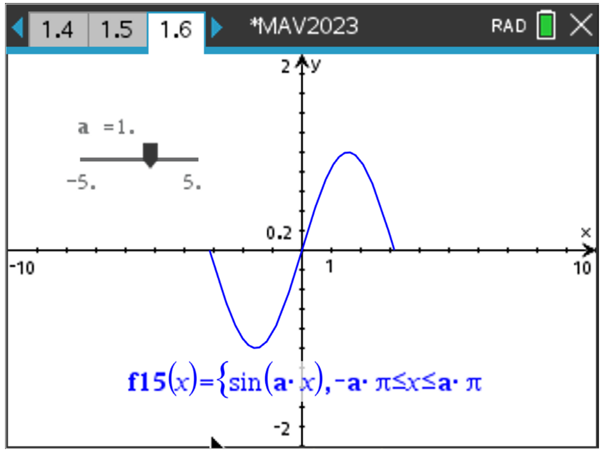
D. $2a$

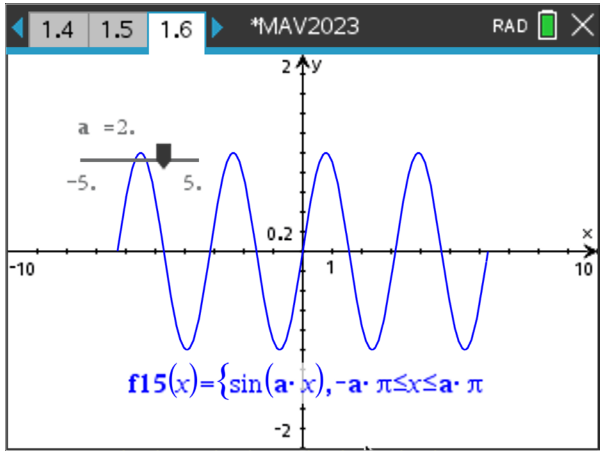
E. a^2

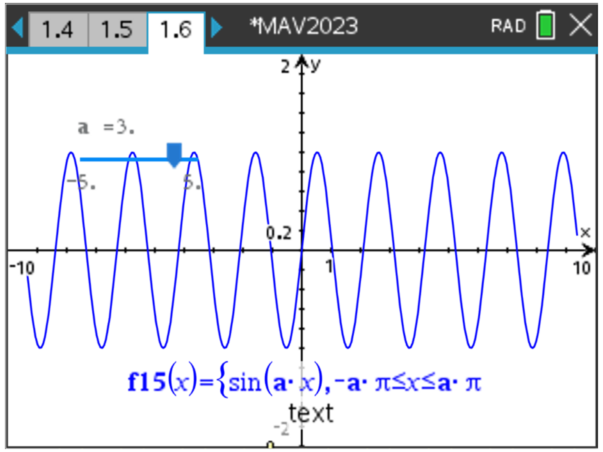
$$f:[-a\pi, a\pi] \rightarrow R, f(x) = \sin(ax)$$

a	Number of local minima
1	1
2	4
3	9

The number of local minima is a^2.







E Question 12 (C 26) 29%

Question 12

The probability mass function for the discrete random variable X is shown below.

X	-1	0	1	2
$\Pr(X = x)$	k^2	$3k$	k	$-k^2 - 4k + 1$

The maximum possible value for the mean of X is:

A. 0

B. $\frac{1}{3}$

C. $\frac{2}{3}$

D. 1

E. 2

From observation $k \geq 0$ and the maximum will occur when $k = 0$, $E(X) = 2$.

$E(X) = -k^2 + k - 2k^2 - 8k + 2$

$E(X) = -3k^2 - 7k + 2$, when $k = 0$, $E(X) = 2$.

D Question 14 (C,B, 22,33) 29%

Question 14

A polynomial has the equation $y = x(3x-1)(x+3)(x+1)$.

The number of tangents to this curve that pass through the positive x-intercept is

A. 0

B. 1

C. 2

D. 3

E. 4

$y = x(3x-1)(x+3)(x+1)$

The positive x-intercept is $\frac{1}{3}$.

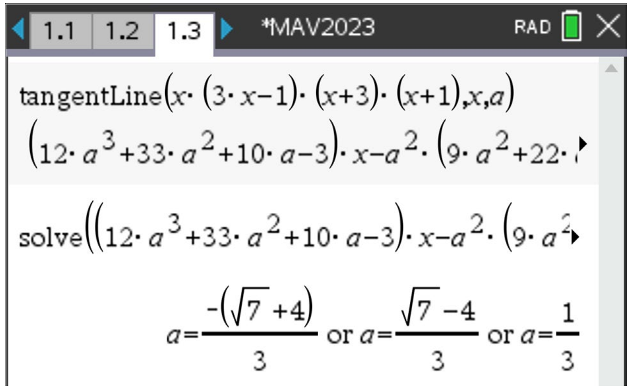
Find the tangent line at $x = a$.

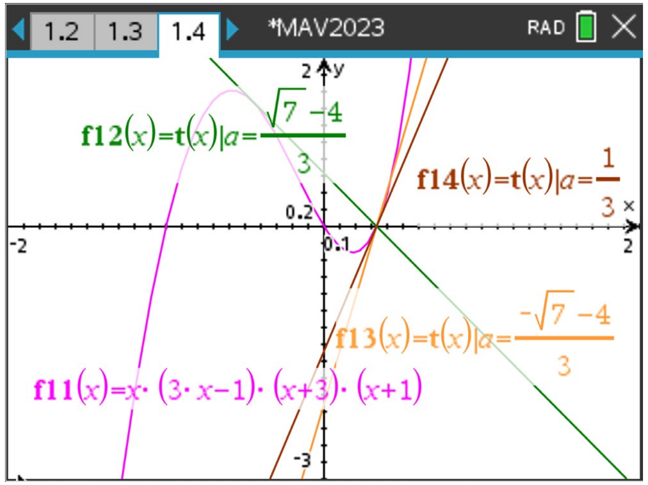
$y_T = \left(12a^3 + 33a^2 + 10a - 3\right)x - a^2\left(9a^2 + 22a + 5\right)$

Solve $y_T\left(\frac{1}{3}\right) = 0$ for a.

$a = \frac{-\sqrt{7}-4}{3}$, $a = \frac{\sqrt{7}-4}{3}$ or $a = \frac{1}{3}$

Hence three solutions.

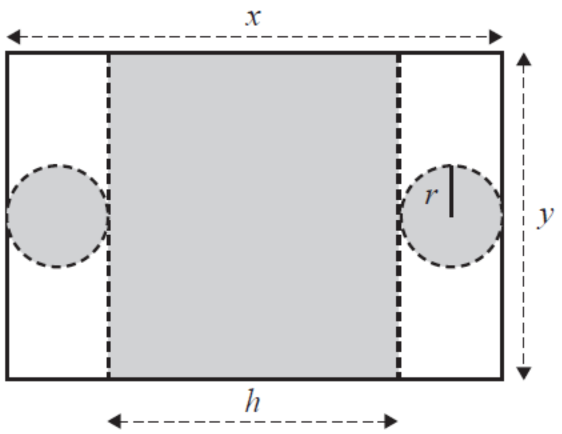




B Question 17 (C,D, 26,26) 28%

Question 17

A cylinder of height h and radius r is formed from a thin rectangular sheet of metal of length x and width y, by cutting along the dashed lines shown below.



The volume of the cylinder, in terms of x and y, is given by

A. $\pi x^2 y$

B. $\dfrac{\pi x y^2 - 2y^3}{4\pi^2}$

C. $\dfrac{2y^3 - \pi x y^2}{4\pi^2}$

D. $\dfrac{\pi x y - 2y^2}{2\pi}$

E. $\dfrac{2y^2 - \pi x y}{2\pi}$

The volume of the cylinder, in terms of x and y, is given by

The base of the cylinder has a circumference of $2\pi r$ units.

Hence $y = 2\pi r$, $r = \dfrac{y}{2\pi}$.

$h = x - 4r$, $h = x - \dfrac{2y}{\pi}$

The formula for volume of the cylinder is $V = \pi r^2 h$.

$$V = \pi\left(\frac{y}{2\pi}\right)^2\left(x - \frac{2y}{\pi}\right) = \frac{\pi x y^2 - 2y^3}{4\pi^2}$$

E Question 11 (D 51) 22%

Question 11

Two functions, f and g, are continuous and differentiable for all $x \in R$. It is given that $f(-2) = -7$, $g(-2) = 8$ and $f'(-2) = 3$, $g'(-2) = 2$.

The gradient of the graph $y = f(x) \times g(x)$ at the point where $x = -2$ is

A. -10

B. -6

C. 0

D. 6

E. 10

$$\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$$

$$= f'(-2)g(-2) + g(-2)f'(-2)$$

$$= 3 \times 8 + -7 \times 2$$

$$= 10$$

Extended Answer

- Give an exact answer unless otherwise stated.

$$\frac{1}{3} = 0.3 \qquad \frac{1}{3} \text{ OR } 0.3$$

- No calculator syntax.
- Show working for questions worth more than one mark. (Rule and answer)
- Work to more decimal places than the required answer.
- Use the variables that are given within the question. (SAC questions)
- If a rule is required, give a rule, not just an expression.

Extended Answer

- Reread questions.
- Take time when drawing graphs – scale axes, check if coordinates are required, one sharp line…
- Don't assume steps in *Show That* questions.
- Use brackets correctly.
- Transcribe formulas correctly (reread the calculator).
- Put units in the final answer.
- Check that the final answer makes sense.
- Use the calculator….check the entry
- Check the float on the calculator.

Extended Answer

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- Be careful with hand writing.
- Use a horizontal vinculum with fractions.
- HB pencil or darker, no light pens….
- Check intervals $[-2,2]$ not $[2,-2]$

Question 1a.

Marks	0	1	Average
%	9	91	0.9

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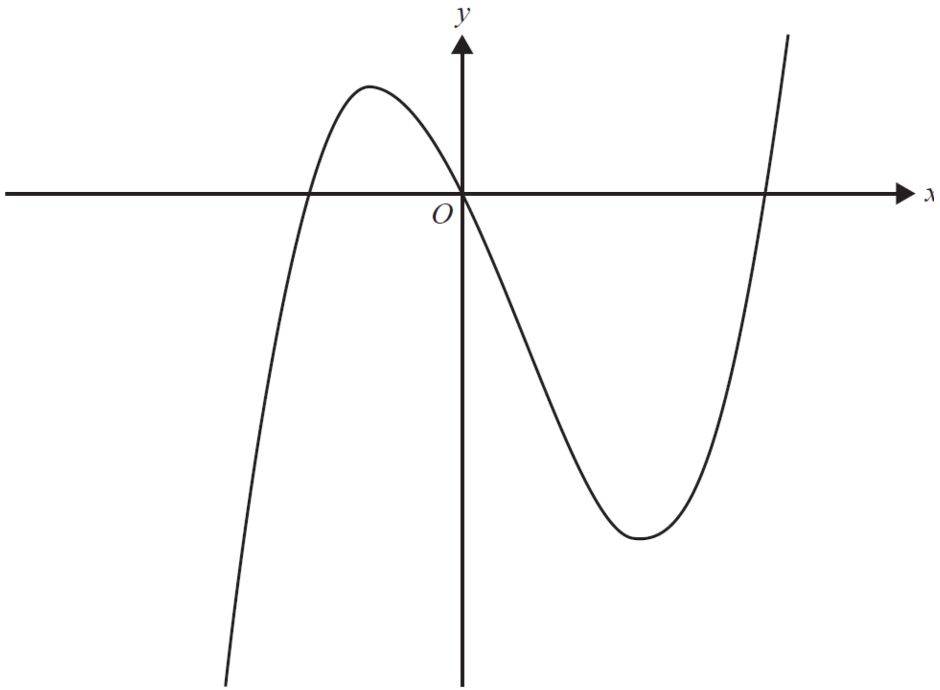
Question 1 (11 marks)

Let $f: R \to R, f(x) = x(x-2)(x+1)$. Part of the graph of f is shown below.



a. State the coordinates of all axial intercepts of f.

Solve $f(x) = 0$ for x.

$x = -1,\ 0$ or 2

The coordinates of the axial intercepts are $(-1,0)$, $(0,0)$ and $(2,0)$.

Question 1b.

Marks	0	1	2	Average
%	6	25	68	1.6

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b. Find the coordinates of the stationary points of f.

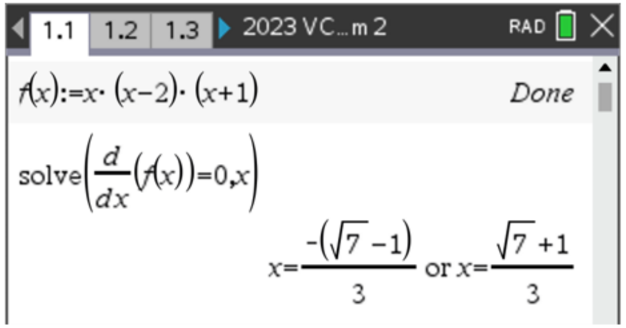
Solve $f'(x) = 0$ for x **OR** Use fmax and fmin

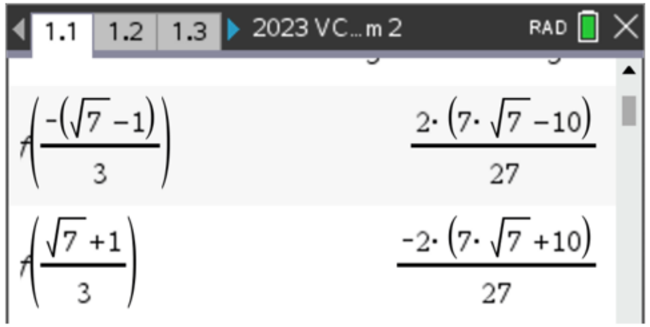
$$x = \frac{-\sqrt{7}+1}{3} \text{ or } x = \frac{\sqrt{7}+1}{3}$$

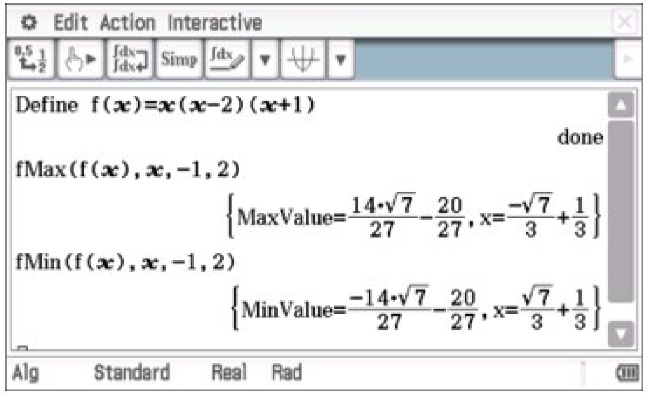
$$f\left(\frac{-\sqrt{7}+1}{3}\right) = \frac{2\left(7\sqrt{7}-10\right)}{27},\ f\left(\frac{\sqrt{7}+1}{3}\right) = \frac{-2\left(7\sqrt{7}+10\right)}{27}$$

The coordinates of the turning points are

$$\left(\frac{-\sqrt{7}+1}{3}, \frac{2\left(7\sqrt{7}-10\right)}{27}\right) \text{ and } \left(\frac{\sqrt{7}+1}{3}, \frac{-2\left(7\sqrt{7}+10\right)}{27}\right).$$







Question 1ci.

Marks	0	1	Average
%	14	86	0.9

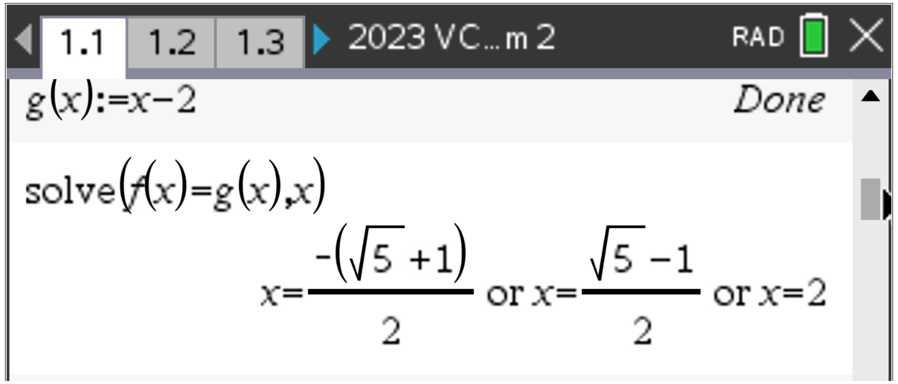
THE MATHEMATICAL ASSOCIATION OF VICTORIA

c. **i.** Let $g : R \to R$, $g(x) = x - 2$.

Find the values of x for which $f(x) = g(x)$.

Solve $f(x) = g(x)$ for x.

$$x = 2,\ x = \frac{-1 \pm \sqrt{5}}{2}$$



Question 1cii.

Marks	0	1	2	Average
%	25	15	61	1.4

THE MATHEMATICAL ASSOCIATION OF VICTORIA

ii. Write down an expression using definite integrals that gives the area of the regions bound by f and g.

$$\text{Area of the bounded region} = \int_{\frac{-\sqrt{5}-1}{2}}^{\frac{\sqrt{5}-1}{2}} \left(f(x)-g(x)\right)dx + \int_{\frac{\sqrt{5}-1}{2}}^{2} \left(g(x)-f(x)\right)dx$$

OR

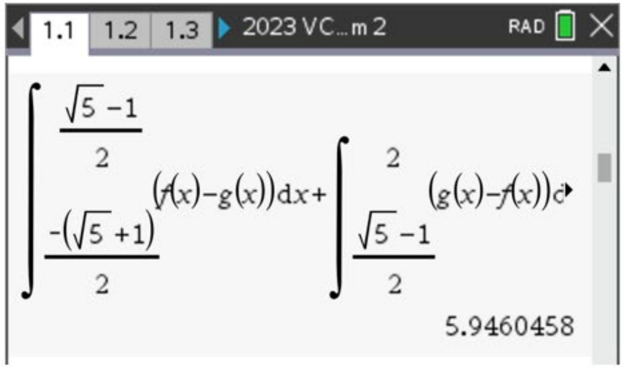
$$\text{Area of the bounded region} = \int_{-\frac{(\sqrt{5}+1)}{2}}^{2} \left|f(x)-g(x)\right|dx$$

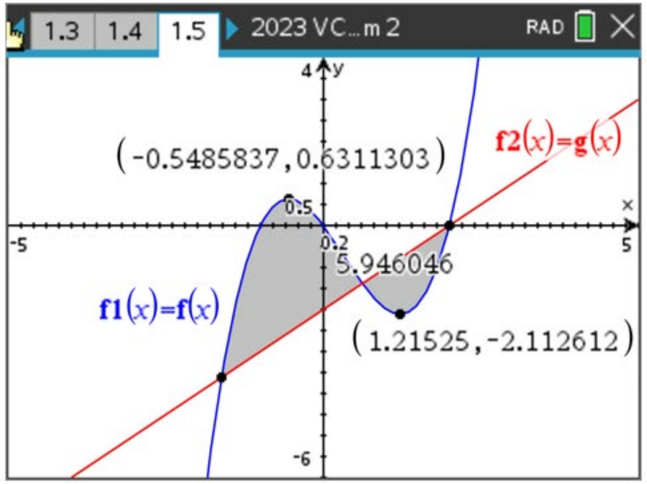
Question 1ciii.

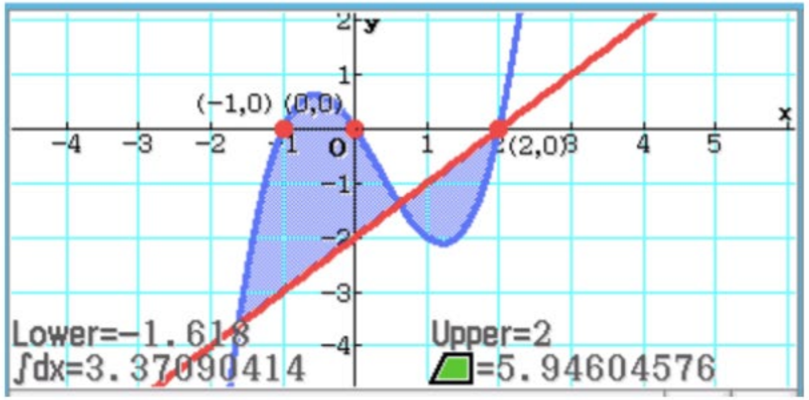
Marks	0	1	Average
%	42	58	0.6

iii. Hence, find the total area of the regions bound by f and g, correct to two decimal places.

Area = 5.95 units^2 correct to two decimal places







Question 1d.

Marks	0	1	2	3	4	Average
%	61	11	9	6	13	1.0

THE MATHEMATICAL ASSOCIATION OF VICTORIA

d. Let $h : R \to R$, $h(x) = (x - a)(x - b)^2$, where $h(x) = f(x) + k$ and $a, b, k \in R$.

Find the possible values of a and b.

Method 1

$(x-a)(x-b)^2 = f(x)+k$

Expand both sides

$x^3 - (a+2b)x^2 + (2ab+b^2)x - ab^2 = x^3 - x^2 - 2x + k$

Equate coefficients

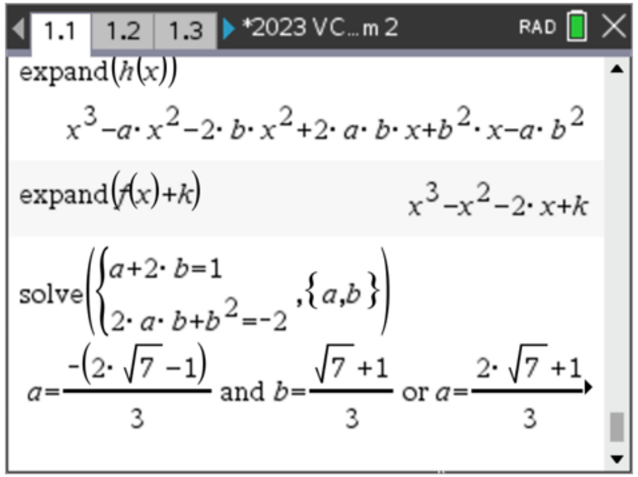
$-(a+2b) = -1$

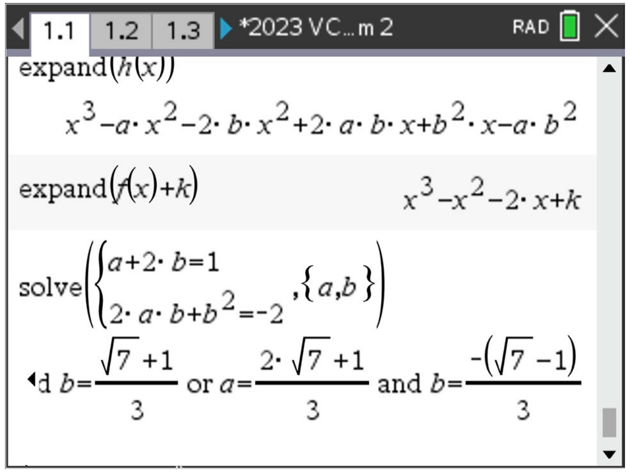
$2ab + b^2 = -2$

$-ab^2 = k$ (optional as k is not asked for)

$b = \dfrac{\sqrt{7}+1}{3}$, $a = \dfrac{-2\sqrt{7}+1}{3}$

$b = \dfrac{-\sqrt{7}+1}{3}$, $a = \dfrac{2\sqrt{7}+1}{3}$





Question 1d.

Marks	0	1	2	3	4	Average
%	61	11	9	6	13	1.0

THE MATHEMATICAL ASSOCIATION OF VICTORIA

Method 2

The turning point is on the x-axis.

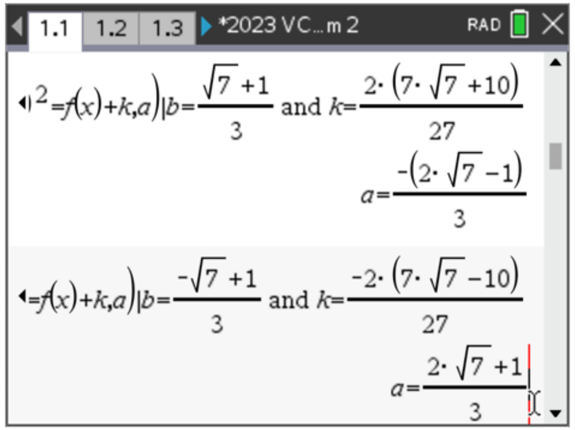
Solve $(x-a)(x-b)^2 = f(x)+k$ for a when

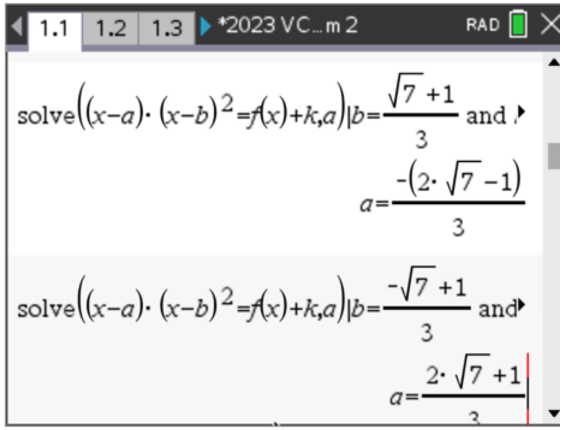
$$k=\frac{2\left(7\sqrt{7}+10\right)}{27} \text{ and } b=\frac{\sqrt{7}+1}{3}$$

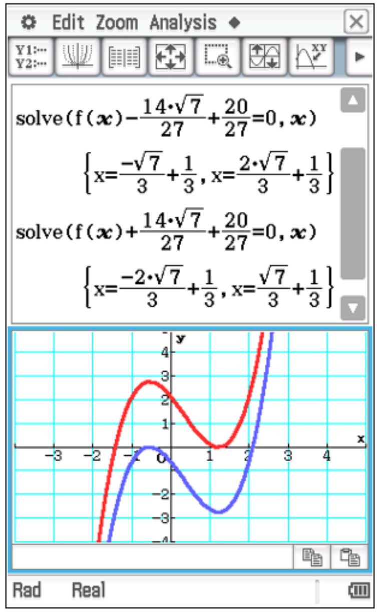
$$k=\frac{-2\left(7\sqrt{7}-10\right)}{27} \text{ and } b=\frac{-\sqrt{7}+1}{3}$$

$$b=\frac{\sqrt{7}+1}{3},\ a=\frac{-2\sqrt{7}+1}{3}$$

$$b=\frac{-\sqrt{7}+1}{3},\ a=\frac{2\sqrt{7}+1}{3}$$







Question 1d.

Marks	0	1	2	3	4	Average
%	61	11	9	6	13	1.0

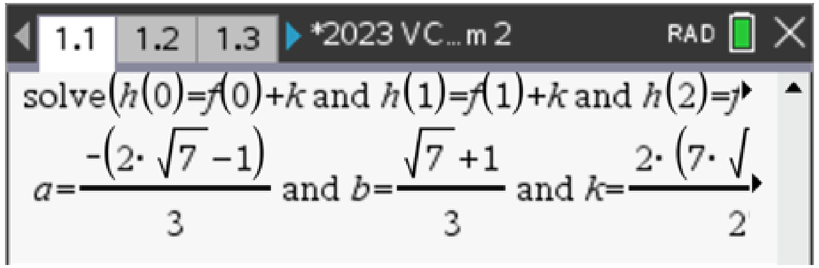
THE MATHEMATICAL ASSOCIATION OF VICTORIA

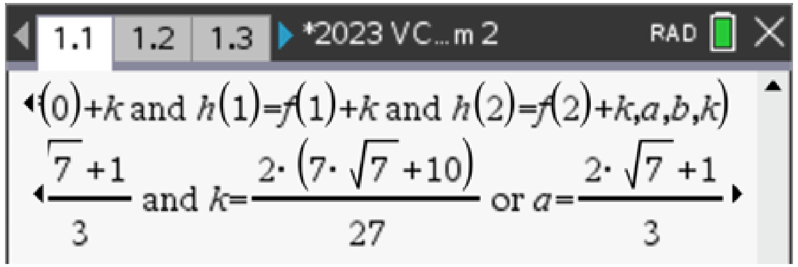
Method 3

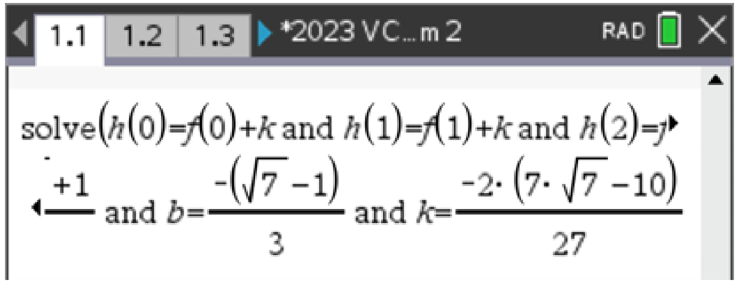
Solve $h(0) = f(0) + k$, $h(1) = f(1) + k$ and $h(2) = f(2) + k$ for a and b

$$b = \frac{\sqrt{7}+1}{3},\ a = \frac{-2\sqrt{7}+1}{3}$$

$$b = \frac{-\sqrt{7}+1}{3},\ a = \frac{2\sqrt{7}+1}{3}$$







Question 2a.

Marks	0	1	2	Average
%	16	18	66	1.5

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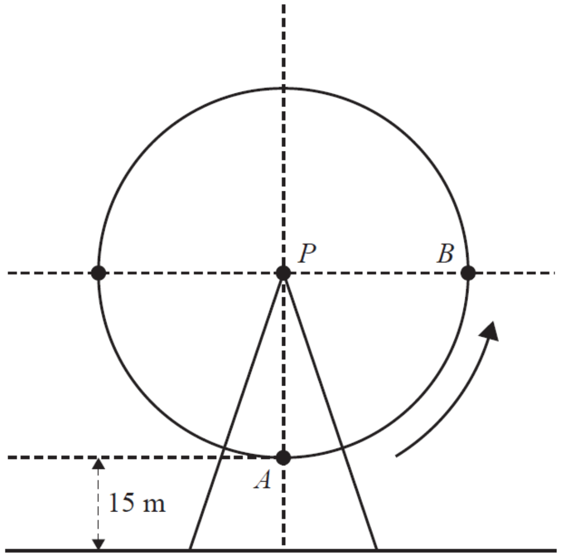
Question 2 (11 marks)

The following diagram represents an observation wheel, with its centre at point *P*. Passengers are seated in pods, which are carried around as the wheel turns. The wheel moves anticlockwise with constant speed and completes one full rotation every 30 minutes. When a pod is at the lowest point of the wheel (point *A*), it is 15 metres above the ground. The wheel has a radius of 60 metres.



Consider the function $h(t) = -60\cos(bt) + c$ for some $b, c \in R$, which models the height above the ground of a pod originally situated at point *A*, after time *t* minutes.

a. Show that $b = \frac{\pi}{15}$ and $c = 75$.

$$\text{The period } = \frac{2\pi}{b} = 30$$

$$b = \frac{2\pi}{30} = \frac{\pi}{15}$$

Solve $h(0) = 15$

$$-60 + c = 15$$

$$c = 75$$

Question 2b.

Marks	0	1	2	Average
%	54	3	42	0.9

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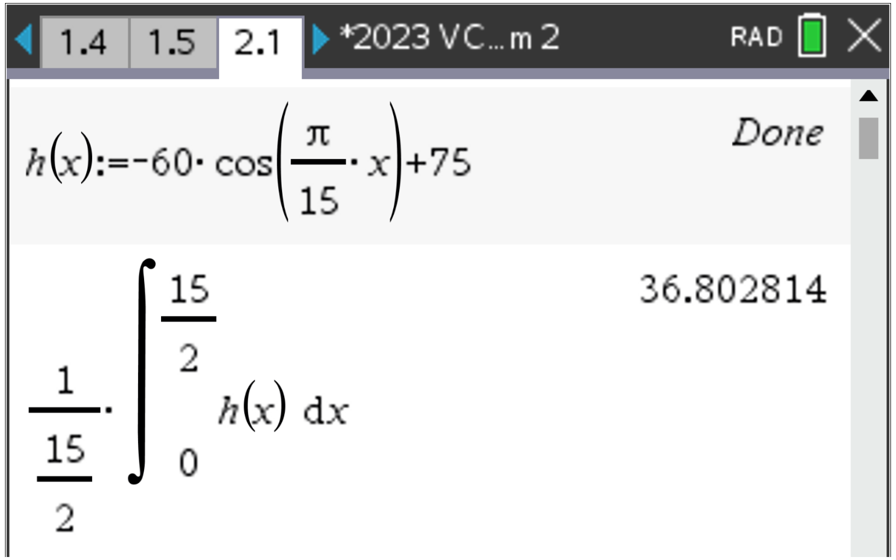
b. Find the average height of a pod on the wheel as it travels from point A to point B. Give your answer in metres, correct to two decimal places.

$$\text{Average height} = \frac{1}{7.5}\int_0^{7.5} h(t)dt$$

$$= 36.80 \text{ m} \quad \text{correct to two decimal places}$$



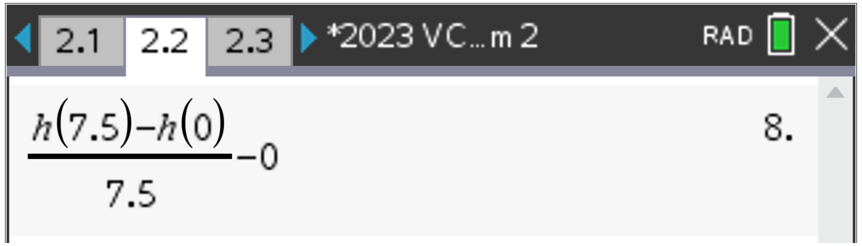
Question 2c.

Marks	0	1	Average
%	45	55	0.5

THE MATHEMATICAL ASSOCIATION OF VICTORIA

c. Find the average rate of change, in metres per minute, of the height of a pod on the wheel as it travels from point A to point B.

$$\text{Average rate of change in height} = \frac{h(7.5) - h(0)}{7.5}$$
$$= 8 \text{ m / min}$$



Question 2di.

Marks	0	1	Average
%	60	40	0.4

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After 15 minutes, the wheel stops moving and remains stationary for 5 minutes. After this, it continues moving at double its previous speed for another 7.5 minutes.

The height above the ground of a pod that was initially at point A, after t minutes, can be modelled by the piecewise function w:

$$w(t) = \begin{cases} h(t) & 0 \le t < 15 \\ k & 15 \le t < 20 \\ h(mt + n) & 20 \le t \le 27.5 \end{cases}$$

where $k \ge 0$, $m \ge 0$ and $n \in R$.

d. **i.** State the values of k and m.

$$k = 75 + 60 = 135$$

$$\text{Period } = \frac{2\pi}{bm} = \frac{30}{m} = \frac{30}{2}$$

$$m = 2$$

Question 2dii.

Marks	0	1	2	Average
%	76	12	12	0.4

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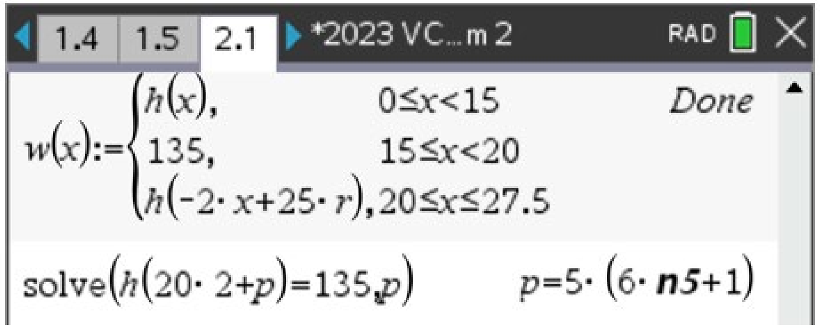
ii. Find **all** possible values of n.

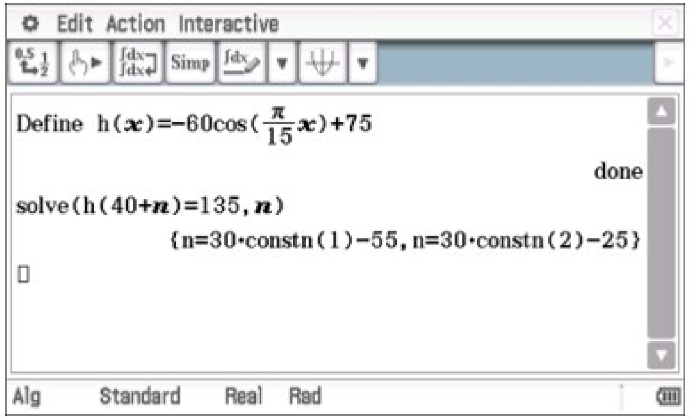
Solve $w(20) = h(2(20)+n) = 135$ or $w(27.5) = 15$

$n = 5 + 30p,\ p \in Z$

OR

$n = 30p - 55,\ p \in Z$ other possibilities

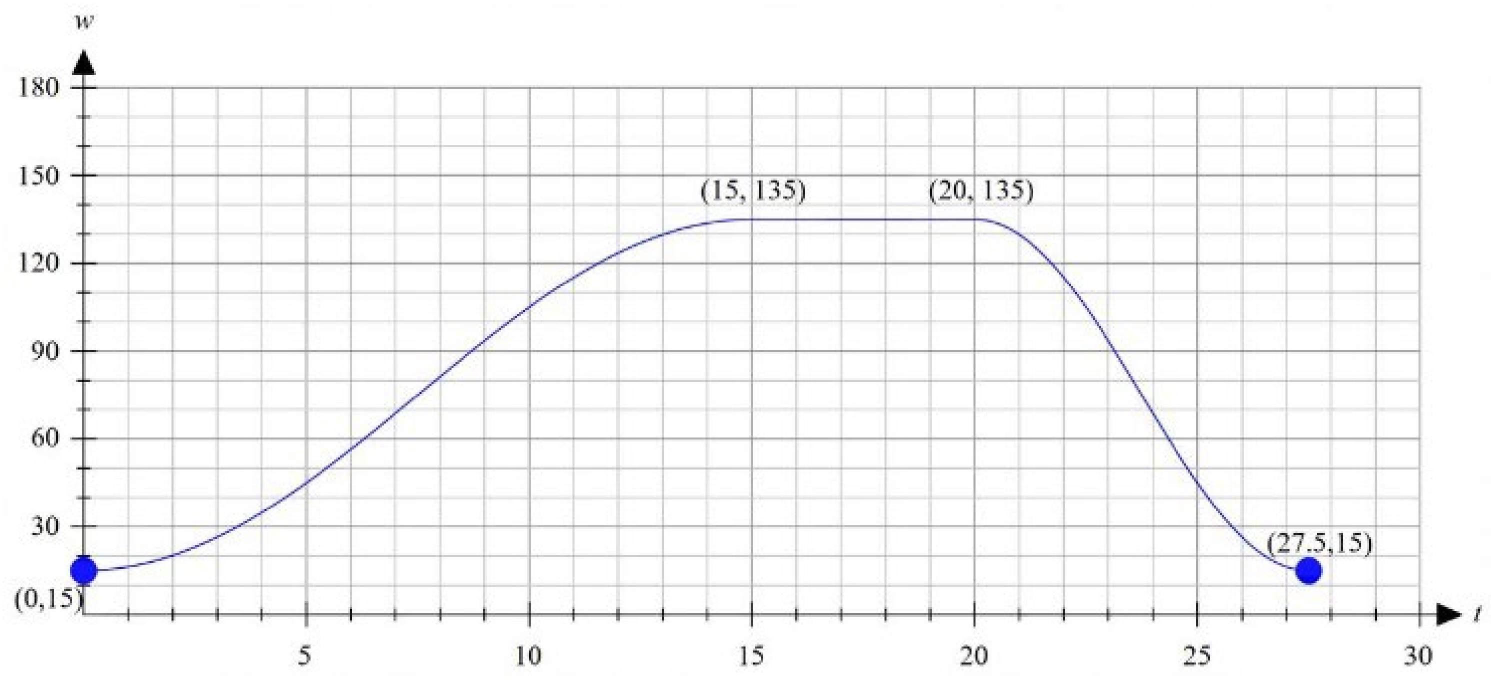




Question 2diii.

Marks	0	1	2	3	Average
%	39	25	12	24	1.2

iii. Sketch the graph of the piecewise function w on the axes below, showing the coordinates of the endpoints.



Question 3a.

Marks	0	1	Average
%	25	75	0.7

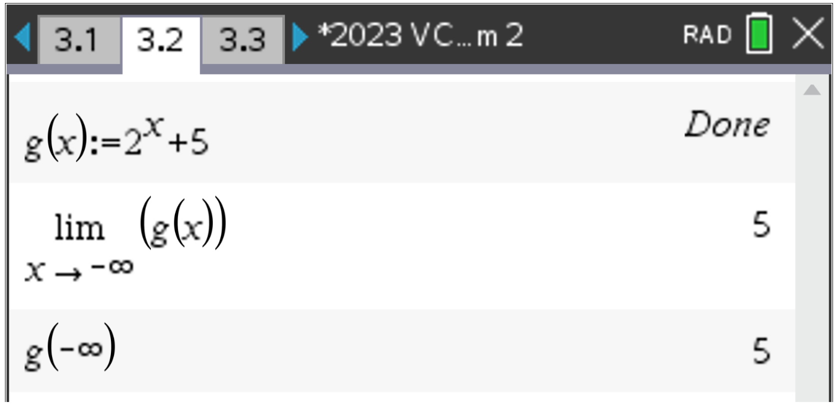
THE MATHEMATICAL ASSOCIATION OF VICTORIA

Question 3 (12 marks)

Consider the function $g : R \to R,\ g(x) = 2^x + 5$.

a. State the value of $\lim\limits_{x \to -\infty} g(x)$.

$$\lim_{x \to -\infty} g(x) = 5$$



Question 3b.

Marks	0	1	Average
%	15	85	0.9

b. The derivative, $g'(x)$, can be expressed in the form $g'(x) = k \times 2^x$.

Find the real number k.

$$g'(x) = k \times 2^x,\ g'(x) = \log_e(2) \times 2^x$$

$$k = \log_e(2) \ \textbf{OR} \ k = \ln(2)$$

Question 3ci.

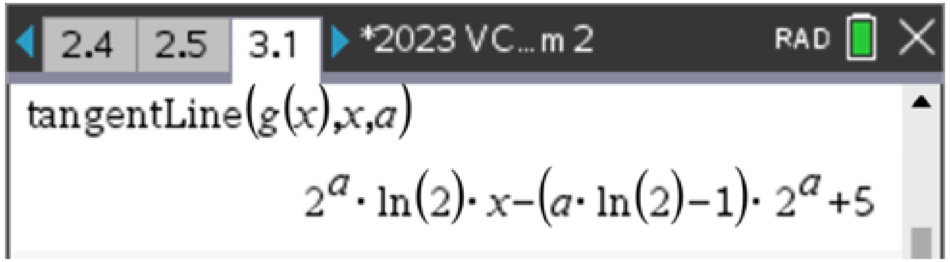
Marks	0	1	Average
%	48	52	0.5

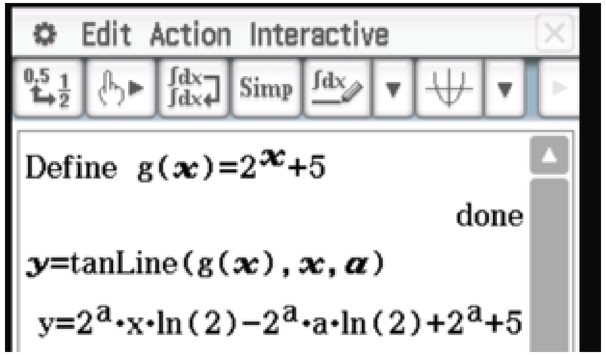
THE MATHEMATICAL ASSOCIATION OF VICTORIA

c. i. Let a be a real number. Find, in terms of a, the equation of the tangent to g at the point $(a, g(a))$.

$$y = 2^a \log_e(2)x - (a\log_e(2) - 1) \times 2^a + 5$$

OR $y = 2^a \log_e(2)x - a2^a \log_e(2) + 2^a + 5$





Question 3cii.

Marks	0	1	2	Average
%	52	34	15	0.6

THE MATHEMATICAL ASSOCIATION OF VICTORIA

ii. Hence, or otherwise, find the equation of the tangent to g that passes through the origin, correct to three decimal places.

Substitute (0, 0) into $y = 2^a \log_e(2)x - (a\log_e(2) - 1) \times 2^a + 5$ and solve for a.

$a = 2.61784...$

$y = 4.255x$ correct to three decimal places

2.4 | 2.5 | 3.1 | *2023 VC...m 2 | RAD

$\text{solve}\left(2^a \cdot \ln(2) \cdot x - (a \cdot \ln(2) - 1) \cdot 2^a + 5 = 0, a\right)$

$a = 2.6178471$

$2^a \cdot \ln(2) \cdot x - (a \cdot \ln(2) - 1) \cdot 2^a + 5 | a = 2.6178470$

$4.2547687 \cdot x - 6.\text{E}{-13}$

$\text{solve}(-2^a \cdot a \cdot \ln(2) + 2^a + 5 = 0, a)$

$\{a = 2.617847065\}$

tanLine(g(x), x, 2.617847065)

$4.25476866 \cdot x - 0.00000000337$

Question 3d.

Marks	0	1	Average
%	42	58	0.6

THE MATHEMATICAL ASSOCIATION OF VICTORIA

Let $h : R \rightarrow R, h(x) = 2^x - x^2$.

d. Find the coordinates of the point of inflection for h, correct to two decimal places.

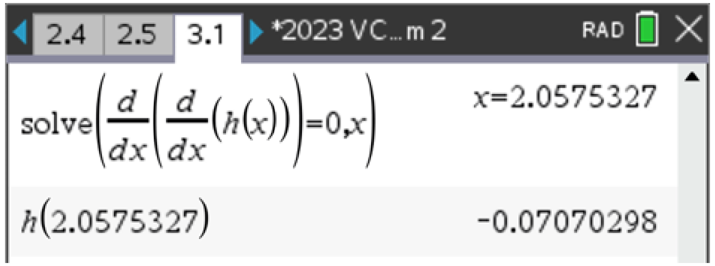
Solve $h''(x) = 0$ for x to find the point of inflection.

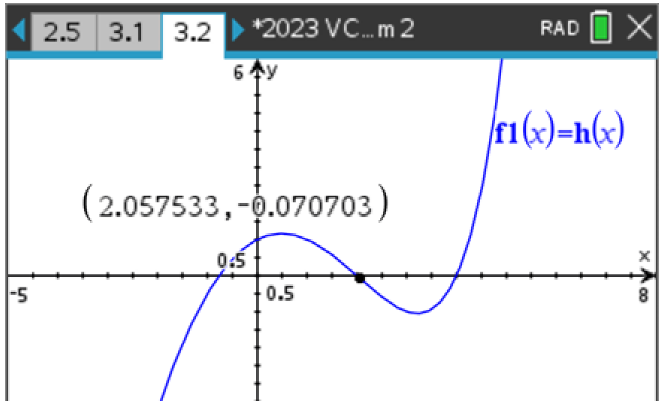
$h(2.057...) = -0.070...$

$(2.06, -0.07)$ correct to two decimal places

OR

Find graphically.





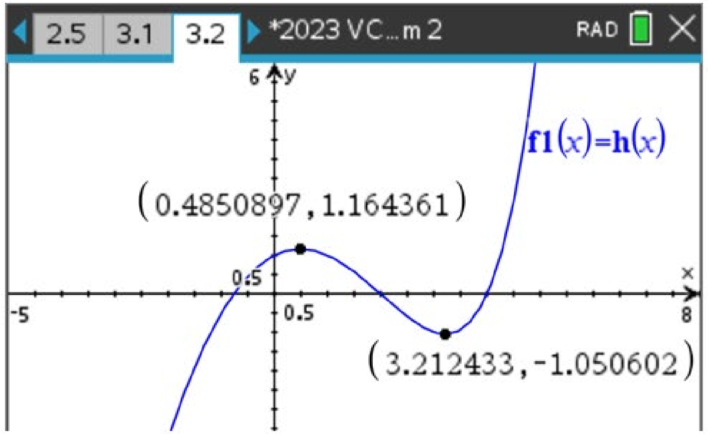
Question 3e.

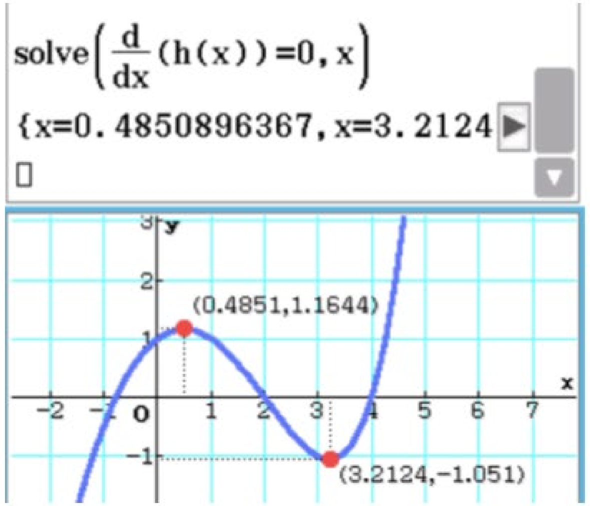
Marks	0	1	Average
%	65	35	0.4

e. Find the largest interval of x values for which h is strictly decreasing.
Give your answer correct to two decimal places.

Solve $\frac{d}{dx}(h(x)) = 0$ for x **OR** use the graph.

The graph is strictly decreasing over the interval $[0.49, 3.21]$, correct to two decimal places.





Question 3f.

Marks	0	1	2	Average
%	36	10	54	1.2

THE MATHEMATICAL ASSOCIATION OF VICTORIA

f. Apply Newton's method, with an initial estimate of $x_0 = 0$, to find an approximate x-intercept of h. Write the estimates x_1, x_2 and x_3 in the table below, correct to three decimal places.

x_0	0
x_1	
x_2	
x_3	

Newton's method $x_{n+1} = x_n - \dfrac{h(x)}{h'(x)}$

Answers are correct to three decimal places.

$x_1 = -1.443$

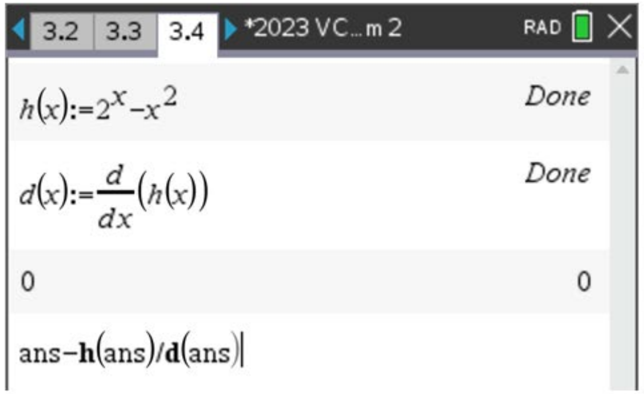
$x_2 = -0.897$

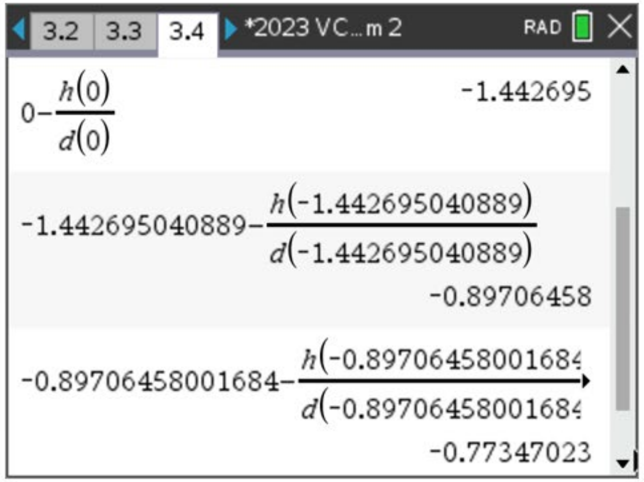
$x_3 = -0.773$

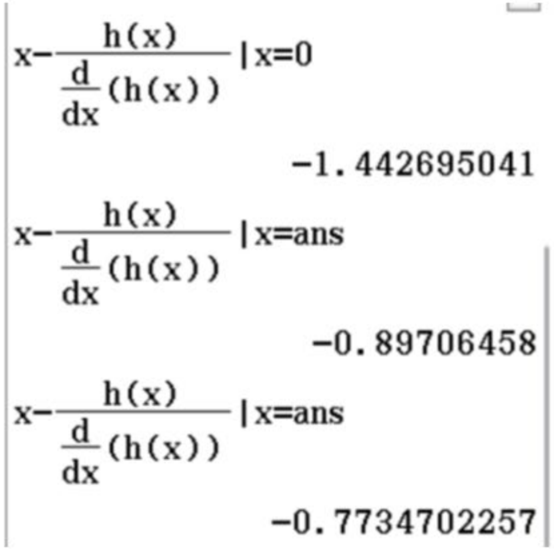
Question 3f.

Marks	0	1	2	Average
%	36	10	54	1.2

THE MATHEMATICAL ASSOCIATION OF VICTORIA







Question 3g.

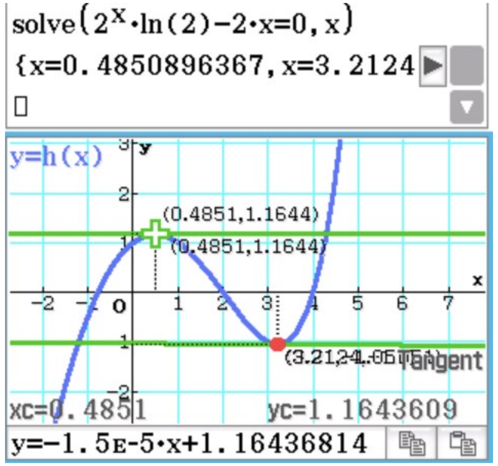
Marks	0	1	Average
%	79	21	0.2

THE MATHEMATICAL ASSOCIATION OF VICTORIA

g. For the function h, explain why a solution to the equation $\log_e(2) \times (2^x) - 2x = 0$ should not be used as an initial estimate x_0 in Newton's method.

The solutions to $\log_e(2) \times 2^x - 2x = 0$ will give the x values of the turning points of the graph.

The tangents to the graph will be horizontal lines and $h'(x) = 0$. $x_{n+1} = x_n - \dfrac{h(x)}{h'(x)}$ will be undefined.



Question 3h.

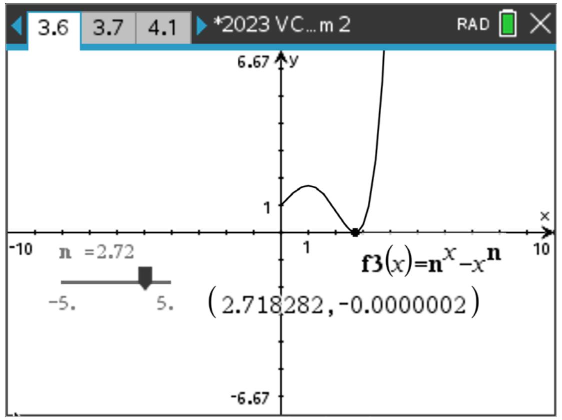
Marks	0	1	2	Average
%	86	12	3	0.2

THE MATHEMATICAL ASSOCIATION OF VICTORIA

h. There is a positive real number n for which the function $f(x) = n^x - x^n$ has a local minimum on the x-axis.

Find this value of n.

Use the slider



OR

Solve $f(x) = 0$ and $f'(x) = 0$.

$f(x) = x^n - n^x = 0$ and $f'(x) = \log_e(n)n^x - nx^{n-1} = 0$

$x^n = n^x$, $x = n$

Hence, $f'(n) = \log_e(n)n^n - nn^{n-1} = 0$.

$n^n(\log_e(n) - 1) = 0$

$\log_e(n) - 1 = 0$

$\log_e(n) = 1$

$n = e$

Define $f(x)=n^x-x^n$

done

$\left\{\begin{array}{l} f(x)=0 \\ \frac{d}{dx}(f(x))=0 \end{array}\right| n, x$

$\{n^x-x^n=0, n^x \cdot \ln(n)-n \cdot x^{n-1}=0\}$

$\text{solve}\left(f\left(\frac{n}{\ln(n)}\right)=0, n\right)$

$\{n=2.718281828\}$

Question 4a.

Marks	0	1	Average
%	21	79	0.8

THE MATHEMATICAL ASSOCIATION OF VICTORIA

Question 4 (15 marks)

A manufacturer produces tennis balls.

The diameter of the tennis balls is a normally distributed random variable D, which has a mean of 6.7 cm and a standard deviation of 0.1 cm.

a. Find $\Pr(D > 6.8)$, correct to four decimal places.

$$X \sim \mathrm{N}\left(6.8, 0.1^2\right)$$

$\Pr\left(D > 6.8\right) = 0.1587$ correct to four decimal places

3.7 | 4.1 | 4.2 | *2023 VC...m 2 | RAD

normCdf(6.8,∞,6.7,0.1) 0.15865526

normCDf(6.8,∞,0.1,6.7)

0.1586552539

Question 4b.

Marks	0	1	Average
%	41	59	0.6

THE MATHEMATICAL ASSOCIATION OF VICTORIA

b. Find the minimum diameter of a tennis ball that is larger than 90% of all tennis balls produced.

Give your answer in centimetres, correct to two decimal places.

$$\Pr(D < d_{\text{min}}) = 0.9$$

$$d_{\text{min}} = 6.83 \text{ correct to two decimal places}$$

```
4.1  4.2  4.3  *2023 VC...m 2                RAD
invNorm(0.9,6.7,0.1)                  6.8281552
```

```
invNormCDf("L",0.9,0.1,6.7)
                     6.828155157
```

Question 4c.

Marks	0	1	Average
%	23	77	0.8

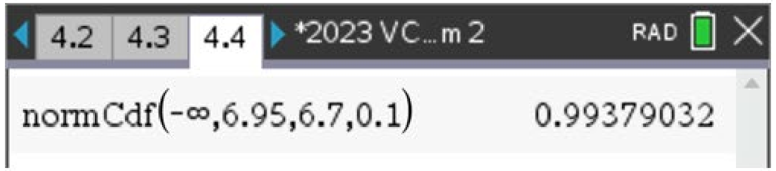
THE MATHEMATICAL ASSOCIATION OF VICTORIA

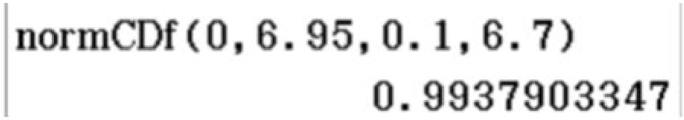
Tennis balls are packed and sold in cylindrical containers. A tennis ball can fit through the opening at the top of the container if its diameter is smaller than 6.95 cm.

c. Find the probability that a randomly selected tennis ball can fit through the opening at the top of the container.

Give your answer correct to four decimal places.

$\Pr(D < 6.95) = 0.9938$ correct to four decimal places





Question 4d.

Marks	0	1	2	Average
%	28	18	54	1.3

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d. In a random selection of 4 tennis balls, find the probability that at least 3 balls can fit through the opening at the top of the container.

Give your answer correct to four decimal places.

$X \sim \text{Bi}(4, 0.99379...)$

$\Pr(X \geq 3) = 0.9998$ correct to four decimal places

4.2 4.3 4.4 *2023 VC...m 2 RAD

binomCdf(4,0.99379032,3,4) 0.99977055

binomialCDf(3,4,4,0.9937903347)

0.9997705514

Question 4e.

Marks	0	1	2	Average
%	31	16	53	1.2

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A tennis ball is classed as grade A if its diameter is between 6.54 cm and 6.86 cm, otherwise it is classed as grade B.

e. Given that a tennis ball can fit through the opening at the top of the container, find the probability that it is classed as grade A.

Give your answer correct to four decimal places.

$$\Pr(6.54 < D < 6.86 \mid D < 6.95)$$

$$= \frac{\Pr(6.54 < D < 6.86)}{\Pr(D < 6.95)}$$

$$= \frac{0.89040...}{0.99379...}$$

$= 0.8960$ correct to four decimal places

4.3 4.4 4.5 *2023 VC...m 2 RAD

$\text{normCdf}(6.54,6.86,6.7,0.1)$ 0.89040142

$\frac{0.89040142120933}{0.99379032014651}$ 0.89596508

$$\frac{\text{normCDf}(6.54, 6.86, 0.1, 6.7)}{\text{normCDf}(0, 6.95, 0.1, 6.7)}$$

0.8959650598

Question 4f.

Marks	0	1	2	Average
%	59	15	26	0.7

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f. The manufacturer would like to improve processes to ensure that more than 99% of all tennis balls produced are classed as grade A.

Assuming that the mean diameter of the tennis balls remains the same, find the required standard deviation of the diameter, in centimetres, correct to two decimal places.

$X \sim \text{N}\left(6.7, \sigma^2\right)$

$\Pr\left(6.54 < D < 6.86\right) \geq 0.99$

$\dfrac{6.86 - 6.7}{\sigma} = 2.5758\ldots$ **OR** $\dfrac{6.54 - 6.7}{\sigma} = -2.5758\ldots$

$0 < \sigma \leq 0.062\ldots$ as the question did not ask for the maximum value of σ.

So $\sigma = 0.00$ or $\sigma = 0.01$ or $\sigma = 0.02$ or $\sigma = 0.03$ or $\sigma = 0.04$ or $\sigma = 0.05$ or $\sigma = 0.06$ correct to two decimal places.

$\text{solve}\left(\text{normCdf}\left(6.54, 6.86, 6.7, s\right) = 0.99, s\right) | s > 0$

$s = 0.062116$

4.1 4.2 4.3 *2023 VC...m 2 RAD

$\text{invNorm}\left(0.995, 0, 1\right)$ 2.5758293

$\text{solve}\left(\dfrac{6.86 - 6.7}{a} = 2.5758293030016, a\right)$

$a = 0.06211592$

```
solve(normCDf(6.54,6.86,x,6.7)=0.99,x)
                    {x=0.0621159173}
normCDf(6.54,6.86,0.06,6.7)
                    0.9923392389
```

Question 4g.

Marks	0	1	2	Average
%	66	15	19	0.5

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g. An inspector takes a random sample of 32 tennis balls from the manufacturer and determines a confidence interval for the population proportion of grade A balls produced.

The confidence interval is (0.7382, 0.9493), correct to 4 decimal places.

Find the level of confidence that the population proportion of grade A balls is within the interval, as a percentage correct to the nearest integer.

$$\hat{p} = \frac{0.7382 + 0.9493}{2} = 0.84375$$

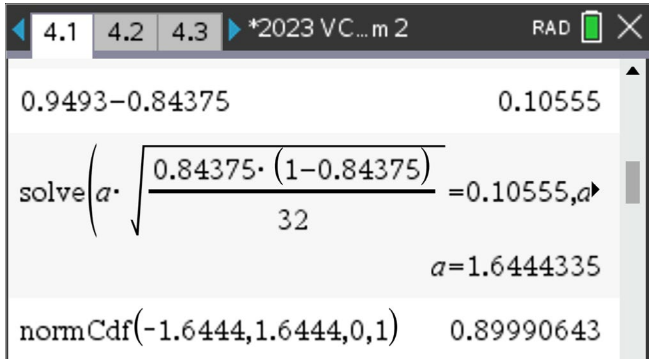
Solve $\text{ME} = z\sqrt{\dfrac{0.84375 \times (1 - 0.84375)}{32}} = 0.9493 - 0.84375 = 0.10555$ for z.

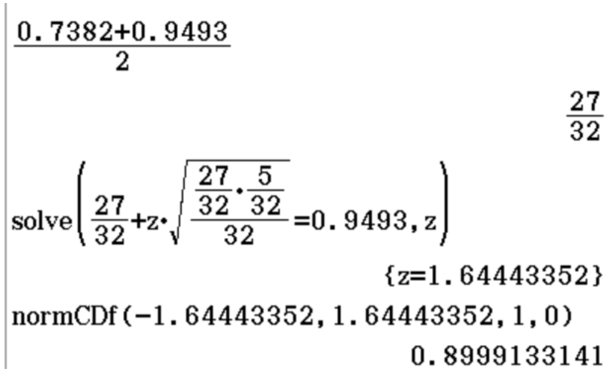
$z = 1.6444...$

$\Pr(-1.6444... < Z < 1.6444...)$

$= 0.8999...$

$= 90\%$ as a percentage correct to the nearest integer





Question 4h.

Marks	0	1	Average
%	43	57	0.6

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A tennis coach uses both grade A and grade B balls. The serving speed, in metres per second, of a grade A ball is a continuous random variable, V, with the probability density function

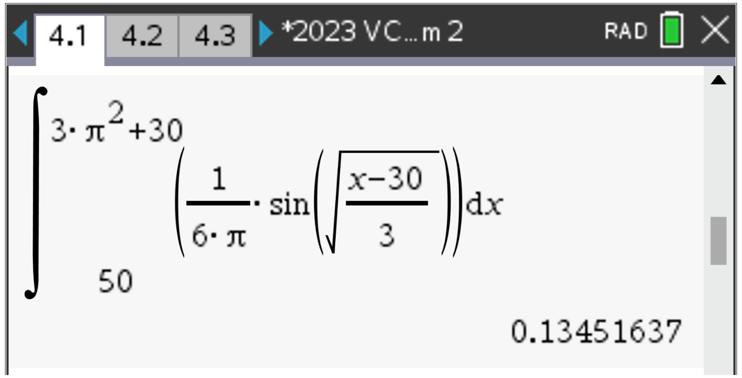
$$f(v) = \begin{cases} \dfrac{1}{6\pi}\sin\left(\sqrt{\dfrac{v-30}{3}}\right) & 30 \le v \le 3\pi^2 + 30 \\ 0 & \text{elsewhere} \end{cases}$$

h. Find the probability that the serving speed of a grade A ball exceeds 50 metres per second.

Give your answer correct to four decimal places.

$$\Pr(V > 50) = \int_{50}^{3\pi^2+30} f(v)\,dv$$

$$= 0.1345 \text{ correct to four decimal places}$$



Question 4i.

Marks	0	1	Average
%	45	55	0.6

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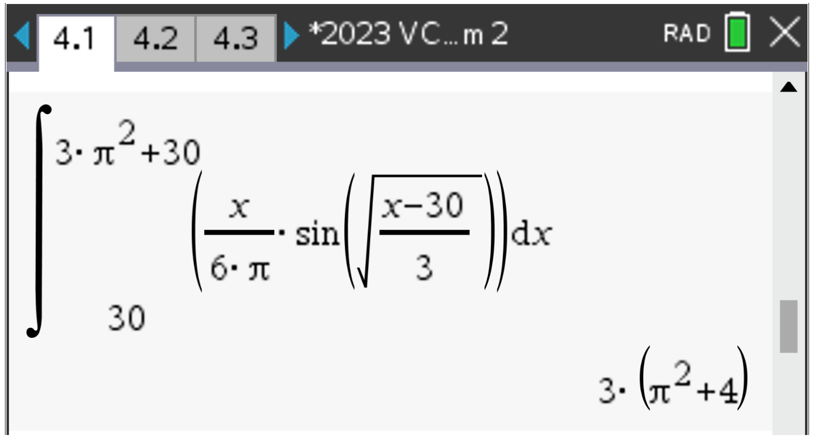
i. Find the **exact** mean serving speed for grade A balls, in metres per second.

$$\mathrm{E}(V) = \int_{30}^{3\pi^2+30} \left(v \times f(v)\right) dv$$

$$= 3\left(\pi^2 + 4\right) = 3\pi^2 + 12$$



Question 4j.

Marks	0	1	2	Average
%	89	5	6	0.2

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The serving speed of a grade B ball is given by a continuous random variable, W, with the probability density function $g(w)$.

A transformation maps the graph of f to the graph of g, where $g(w) = af\left(\frac{w}{b}\right)$.

j. If the mean serving speed for a grade B ball is $2\pi^2 + 8$ m per second, find the values of a and b.

Method 1 (transform the mean)

To maintain an area of 1, $a = \frac{1}{b}$ **OR** $b = \frac{\text{E}(W)}{\text{E}(V)}$

$a = \frac{3}{2},\ b = \frac{2}{3}$

Question 4j.

Marks	0	1	2	Average
%	89	5	6	0.2

THE MATHEMATICAL ASSOCIATION OF VICTORIA

The serving speed of a grade B ball is given by a continuous random variable, W, with the probability density function $g(w)$.

A transformation maps the graph of f to the graph of g, where $g(w) = af\left(\frac{w}{b}\right)$.

j. If the mean serving speed for a grade B ball is $2\pi^2 + 8$ m per second, find the values of a and b.

Method 2 (simultaneous equations)

$$\int_{30b}^{(3\pi^2+30)b} g(w)\,dw = 1 \text{ and } \int_{30b}^{(3\pi^2+30)b} (w \times g(w))\,dw = 2\pi^2 + 8$$

$$a = \frac{3}{2},\ b = \frac{2}{3}$$

4.1 4.2 4.3 *2023 VC...m 2 RAD

$$\text{solve}\left(\int_{30\cdot b}^{(3\cdot\pi^2+30)\cdot b} \left(a\cdot f\left(\frac{w}{b}\right)\right)dw=1 \text{ and } \int \ldots\right)$$

$a=1.5$ and $b=0.66666667$

$$\text{solve}((3\cdot\pi^2+12)\cdot b=2\cdot\pi^2+8, b)$$

$$\left\{b=\frac{2}{3}\right\}$$

Question 5a.

Marks	0	1	2	Average
%	18	48	35	1.2

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Question 5 (11 marks)

Let $f : R \to R, f(x) = e^x + e^{-x}$ and $g : R \to R, g(x) = \frac{1}{2} f(2 - x)$.

a. Complete a possible sequence of transformations to map f to g.

- Dilation of factor $\frac{1}{2}$ from the x axis.
- ______________________________
- ______________________________

$g(x) = \frac{1}{2} f(-(x-2))$, f is an even function

- Reflect in the y-axis
- Translate 2 units right (positive x direction)

OR

- Translate 2 units left (negative x direction)
- Reflect in the y-axis

OR

- Translate 2 units to the right (as f is an even function)

Question 5b.

Marks	0	1	2	Average
%	40	21	39	1.0

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Two functions g_1 and g_2 are created, both with the same rule as g but with distinct domains, such that g_1 is strictly increasing and g_2 is strictly decreasing.

Note the question did not ask for the maximal domain.

b. Give the domain and range for the inverse of g_1.

Using the maximal domain where g_1 is strictly increasing

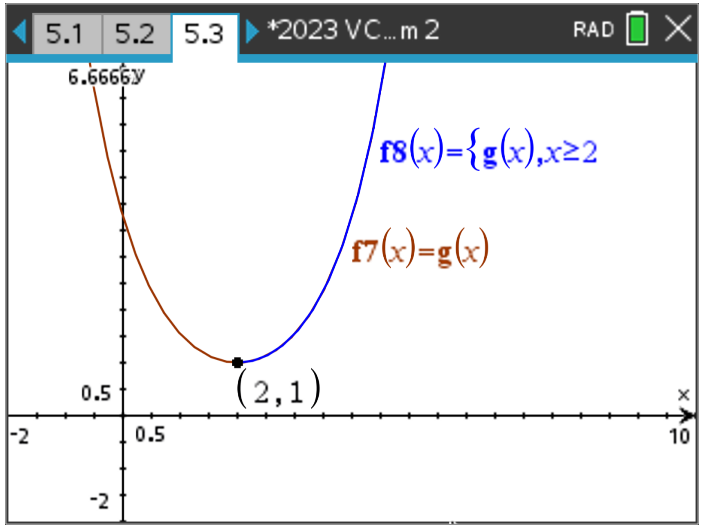
Domain of g_1 is $[2,\infty)$

Range of g_1 is $[1,\infty)$

Hence,

Domain of g_1^{-1} is $[1,\infty)$

Range of g_1^{-1} is $[2,\infty)$



Question 5b.

Marks	0	1	2	Average
%	40	21	39	1.0

THE MATHEMATICAL ASSOCIATION OF VICTORIA

Two functions g_1 and g_2 are created, both with the same rule as g but with distinct domains, such that g_1 is strictly increasing and g_2 is strictly decreasing.

Note the question did not ask for the maximal domain.

b. Give the domain and range for the inverse of g_1.

OR

Using a subset

Domain of g_1^{-1} is $(1, \infty)$

Range of g_1^{-1} is $(2, \infty)$

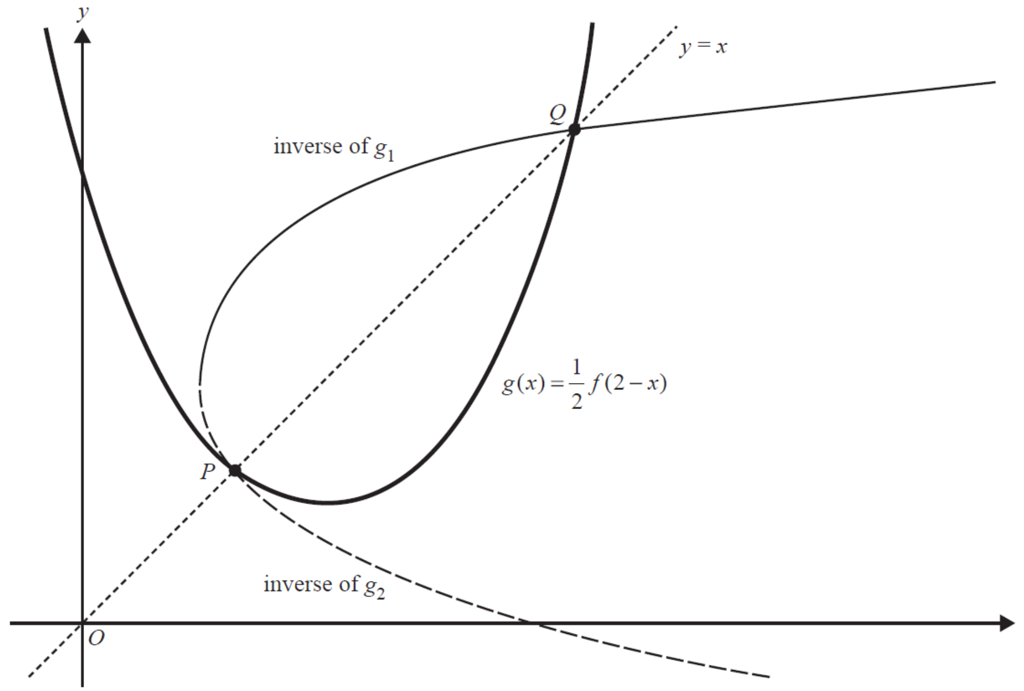
(many possibilities)

If g_1 is defined with a domain which is a subset of $[2, \infty)$ then the domain and range of g_1^{-1} should match this subset.

Question 5ci.

Marks	0	1	Average
%	45	55	0.6

Shown below is the graph of g, the inverses of g_1 and g_2, and the line $y = x$.

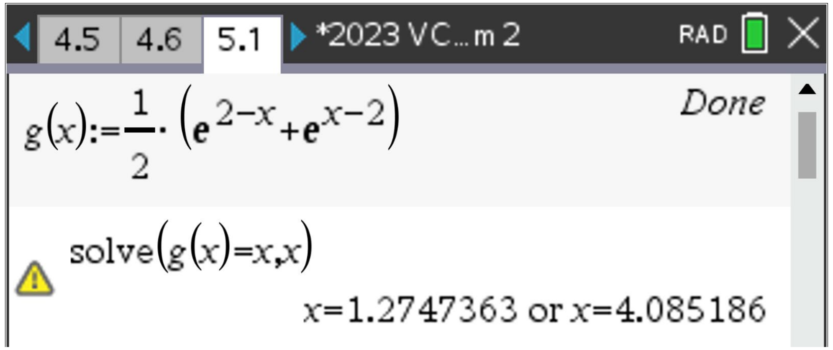


The intersection points between the graphs of $y = x$, $y = g(x)$ and the inverses of g_1 and g_2, are labelled P and Q.

c. **i.** Find the coordinates of P and Q, correct to two decimal places.

Solve $g(x) = x$

P (1.27, 1.27), Q (4.09, 4.09) correct to two decimal places



Question 5cii.

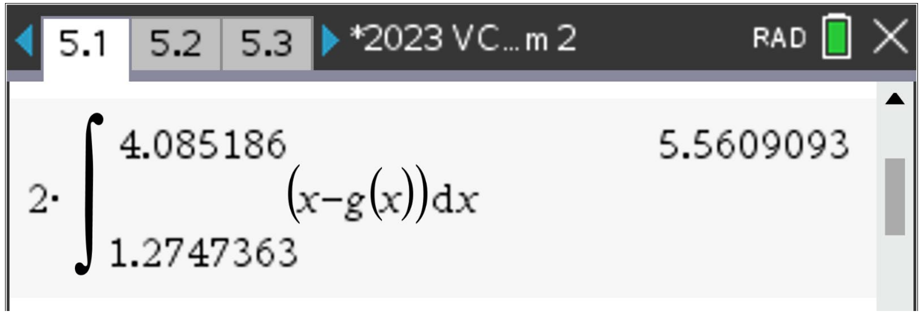
Marks	0	1	2	Average
%	65	6	29	0.6

THE MATHEMATICAL ASSOCIATION OF VICTORIA

ii. Find the area of the region bound by the graphs of g, the inverse of g_1 and the inverse of g_2.
Give your answer correct to two decimal places.

$$2\int_{1.27...}^{4.09...}(x-g(x))dx$$

$= 5.56$ correct to two decimal places



Question 5d.

Marks	0	1	Average
%	87	13	0.1

THE MATHEMATICAL ASSOCIATION OF VICTORIA

Let $h : R \to R, h(x) = \frac{1}{k} f(k - x)$, where $k \in (0, \infty)$.

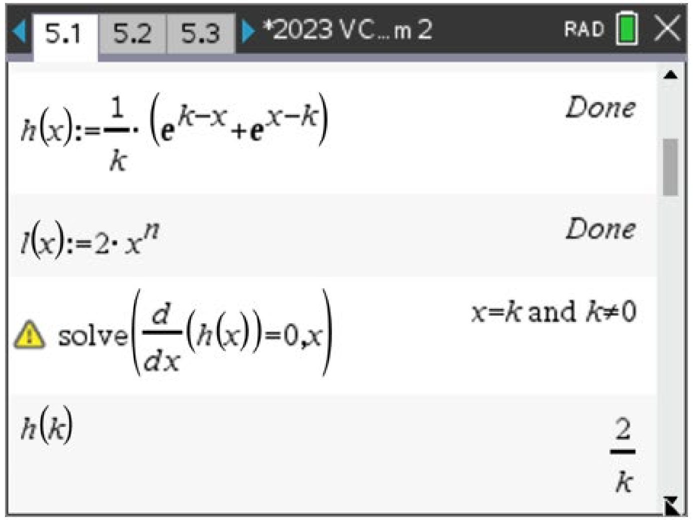
d. The turning point of h always lies on the graph of the function $y = 2x^n$, where n is an integer.

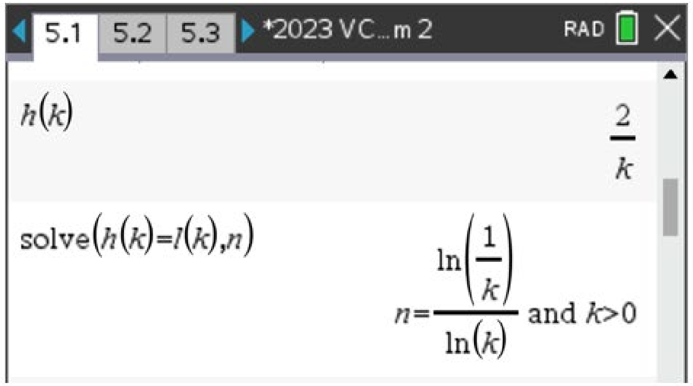
Find the value of n.

$$h'(x) = 0\,,\ \ x = k$$

$$h(k) = \frac{2}{k} = 2k^{-1} = 2k^n$$

$$n = -1$$





Question 5e.

Marks	0	1	Average
%	96	4	0.0

THE MATHEMATICAL ASSOCIATION OF VICTORIA

Let $h_1 : [k, \infty) \to R,\ h_1(x) = h(x)$.

The rule for the **inverse** of h_1 is $y = \log_e\left(\frac{k}{2}x + \frac{1}{2}\sqrt{k^2x^2 - 4}\right) + k$

e. What is the smallest value of k such that h will intersect with the inverse of h_1?
Give your answer correct to two decimal places.

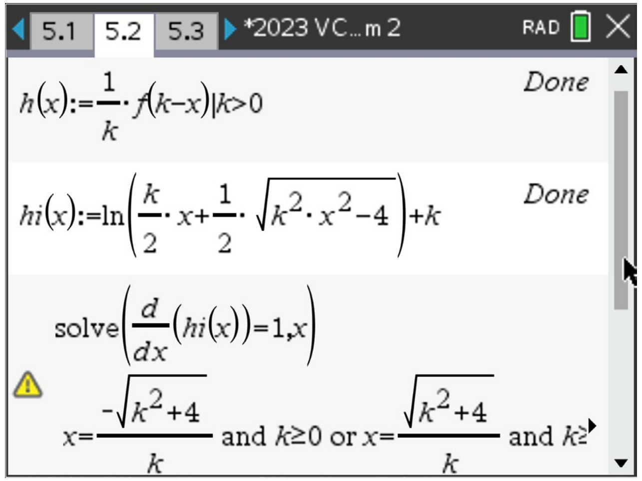
Use the slider

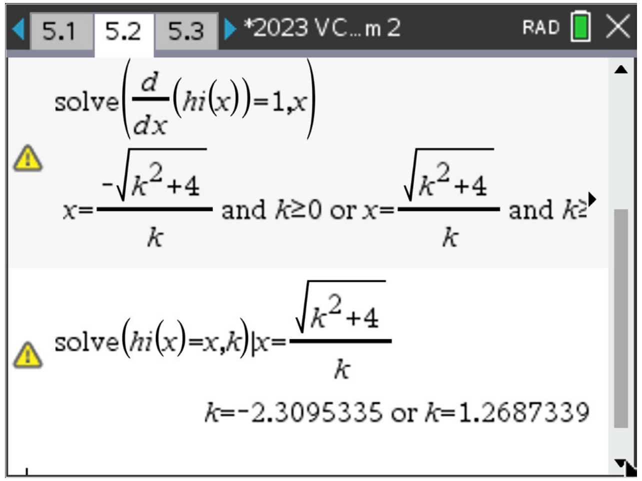
OR

Solve $h_1^{-1\prime}(x) = 1,\ x = \frac{\sqrt{k^2+4}}{k}$

Solve $h_1^{-1}\left(\frac{\sqrt{k^2+4}}{k}\right) = \frac{\sqrt{k^2+4}}{k}$ for k.

$k = 1.27$ correct to two decimal places





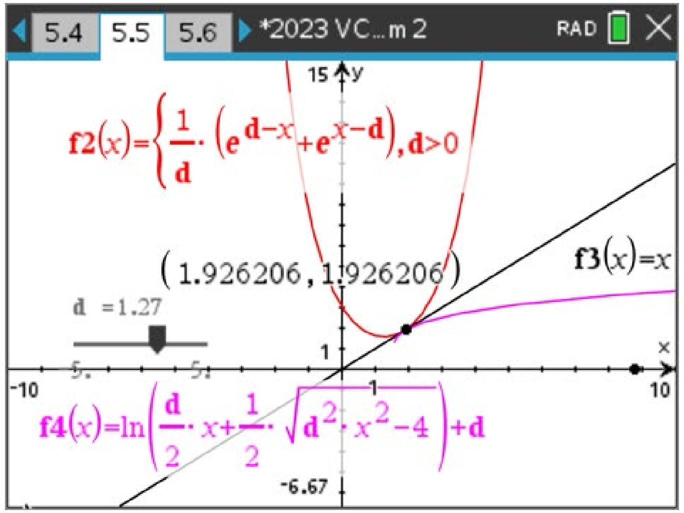
Question 5e.

Marks	0	1	Average
%	96	4	0.0

Let $h_1 : [k, \infty) \to R,\ h_1(x) = h(x)$.

The rule for the **inverse** of h_1 is $y = \log_e\left(\frac{k}{2}x + \frac{1}{2}\sqrt{k^2x^2 - 4}\right) + k$

e. What is the smallest value of k such that h will intersect with the inverse of h_1? Give your answer correct to two decimal places.



Define $f(x)=\ln\left(\frac{k}{2}\cdot x+\frac{1}{2}\cdot\sqrt{k^2\cdot x^2-4}\right)+k$

done

$\left.\begin{cases} f(x)=x \\ \frac{d}{dx}(f(x))=1 \end{cases}\right|_{x,k}$

{x=1.866804, k=1.268733892}

Question 5f.

Marks	0	1	2	Average
%	86	2	12	0.3

THE MATHEMATICAL ASSOCIATION OF VICTORIA

It is possible for the graphs of h and the inverse of h_1 to intersect twice. This occurs when $k = 5$.

f. Find the area of the region bound by the graphs of h and the inverse of h_1, when $k = 5$.
Give your answer correct to two decimal places.

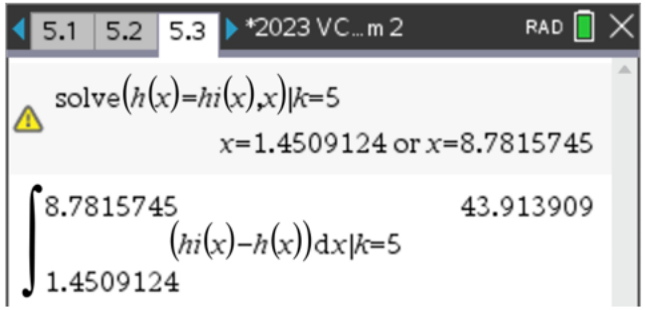
Use bounded area

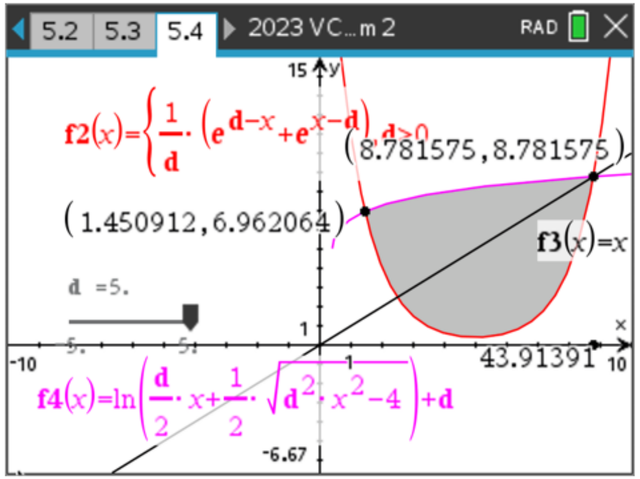
OR

Solve $h_1^{-1}(x) = h(x)$, $x = 1.45\ldots$, $x = 8.78\ldots$

$$A = \int_{1.45091..}^{8.78157...} h_1^{-1}(x) - h(x)\,dx$$

$= 43.91$ correct to two decimal places





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